Experiment Design for Robust Control: Why Do More Work Than Is Needed?

M. Gevers[1], X. Bombois[2], G. Scorletti[3], P. Van den Hof[2] and R. Hildebrand[4]

[1] CESAME, Université Catholique de Louvain, Belgium gevers@csam.ucl.ac.be
[2] Delft Center for Systems and Control, Delft University of Technology, The Netherlands
[3] GREYC Equipe Auto, Caen, France
[4] IMAG, Université J. Fourier, Grenoble, France

1 Introduction

Experiment design for open-loop identification
Optimal input design for system identification was an active area of research in the 1970's, with different quality measures of the identified model being used for this optimal design [1–3]. The questions at that time addressed open-loop identification and the objective functions that were minimized were various measures of the parameter covariance matrix P_θ, where θ is the parameter vector of the model structure.

Let the "true system" be given by:

$$\mathcal{S}: y(t) = \overbrace{G(z,\theta_0)}^{G_0(z)} u(t) + \overbrace{H(z,\theta_0)e(t)}^{v(t)} \tag{1}$$

for some unknown parameter vector $\theta_0 \in \mathbf{R}^k$, where $e(t)$ is white noise of variance σ_e^2, while $G(z,\theta_0)$ and $H(z,\theta_0)$ are stable discrete-time transfer functions, with $H(z,\theta_0)$ a monic and minimum-phase transfer function[5]. In the optimal input design literature, it is assumed that this system is identified with a model structure $\mathcal{M} = \{G(z,\theta),\ H(z,\theta)\}$, $\theta \in \mathbf{R}^k$, that is able to represent the true system; we shall call such structure a "full order model structure".

When Prediction Error identification is used with a full order model structure, the estimated parameter vector $\hat{\theta}_N$ is known to converge, under mild assumptions, to a Gaussian distribution:

$$(\hat{\theta}_N - \theta_0) \overset{N\to\infty}{\longrightarrow} N(0, P_\theta), \tag{2}$$

where the asymptotic parameter covariance matrix P_θ can be estimated from the data. Important examples of optimal design criteria developed in the

[5] By monic is meant that $H(z) = 1 + \sum_{k=1}^{\infty} h_k z^{-k}$.

B.A. Francis et al. (Eds.): Control of Uncertain Systems, LNCIS 329, pp. 139–162, 2006.

1970's are D-optimal design which minimizes $det(P_\theta)$, E-optimal design which minimizes $\lambda_{max}(P_\theta)$, and L-optimal design which minimizes $tr(WP_\theta)$, where W is a nonnegative weighting matrix.

In open-loop identification, the dependence of the covariance matrix on the input spectrum is made apparent by the following expression:

$$P_\theta^{-1} = \left(\frac{N}{\sigma_e^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_u(e^{j\omega},\theta_0)F_u(e^{j\omega},\theta_0)^*\Phi_u(\omega)d\omega\right) + \left(N\frac{1}{2\pi}\int_{-\pi}^{\pi} F_e(e^{j\omega},\theta_0)F_e(e^{j\omega},\theta_0)^* d\omega\right) \tag{3}$$

Here, $F_u(z,\theta_0) = \frac{\Lambda_G(z,\theta_0)}{H(z,\theta_0)}$, $F_e(z,\theta_0) = \frac{\Lambda_H(z,\theta_0)}{H(z,\theta_0)}$, $\Lambda_G(z,\theta) = \frac{\partial G(z,\theta)}{\partial\theta}$ and $\Lambda_H(z,\theta) = \frac{\partial H(z,\theta)}{\partial\theta}$. The formula shows that the data length N and the input spectrum $\Phi_u(\omega)$ appear linearly in the expression of the information matrix P_θ^{-1}, and that, for a given data length N, the input spectrum is the only design quantity that can shape the parameter covariance matrix. Zarrop used Tchebycheff system theory to parametrize the input spectrum in terms of its so-called "trigonometric moments" with respect to the system [2]. These trigonometric moments are defined as $m_k = \frac{1}{2\pi\sigma_e^2}\int_{-\pi}^{\pi}\frac{\Phi_u(\omega)}{|H(e^{j\omega},\theta_0)|^2}cos(k\omega)d\omega$. The information matrix $M_\theta \triangleq P_\theta^{-1}$ can then be expressed as a finite linear combination of these moments, $m_0, m_1, \ldots, m_n$, which express the effect of the filtered input spectrum $\Phi_u(\omega)$ on that information matrix. The number n depends on the degree of $H(z,\theta)$. These moments have to obey some positivity constraint in order for them to be generated by a genuine spectral density function. The optimal input design problem can then be reformulated as a constrained optimization problem in terms of these trigonometric moments. Once the optimal moment sequence has been obtained, it is an easy step to compute a corresponding quasistationary input signal $u(t)$ that will match this optimal moment sequence. A solution can always be obtained using multisines. We refer the reader to [2–5] for background material on Tchebycheff systems, trigonometric moments, and their use in input design problems. One should also note that another way of obtaining a finite parametrization of the information matrix is to restrict the admissible input signals to those that generate a finite dimensional parametrization of the spectrum $\Phi_u(\omega)$, or to approximate the input spectrum by a finite dimensional parametrization: see e.g. [6].

The classical experiment design results of the 1970's were limited to open-loop identification with full order model structures, and they were based on parameter covariance formulas. In the mid-eighties, Ljung and collaborators produced bias and variance formulas [7, 8] directly for the transfer function estimates, rather than for the parameter estimates which only serve as auxiliary variables in the representation of these transfer functions. The asymptotic variance formulas were derived under the assumption that the model order n tends to infinity in some appropriate way when the data length N tends

to infinity. Thus, for the variance of the vector of transfer function estimates $G(z,\hat{\theta}_N),\ H(z,\hat{\theta}_N)$, the following approximation was obtained in [7] under an assumption of high model order:

$$Cov\begin{pmatrix} G(e^{j\omega},\hat{\theta}_N) \\ H(e^{j\omega},\hat{\theta}_N)\end{pmatrix} \cong \frac{n}{N}\Phi_v(\omega)\begin{bmatrix}\Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{eu}(\omega) & \sigma^2\end{bmatrix}^{-1} \tag{4}$$

where n is the model order, N is the number of data, $\Phi_u(\omega)$ is the input spectrum, $\Phi_v(\omega)$ is the output disturbance spectrum, and $\Phi_{ue}(\omega)$ is the cross-spectrum between u and e: see (1). When identification is performed in open loop, we have $\Phi_{ue}(\omega) = 0$. These variance formulas for the transfer function estimates paved the way for the formulation of goal-oriented experiment design problems. In [7] a number of open-loop optimal input design problems were formulated and solved.

Experiment design for closed-loop identification

The first closed-loop optimal design problems, including control-oriented problems, were formulated in [9]. In particular, it was shown in that paper that, when the model is identified for the purpose of designing a minimum variance controller, the optimal design is to perform the experiment in closed loop with the minimum variance controller in the loop. These results were later extended to other control design objectives in [10, 11]. The optimal design criterion used in all these contributions was the average performance degradation, namely the mean squared error between the output of the optimal loop (i.e. the loop that would be obtained if the optimal controller, dependent on the unknown true system, were applied to the system), and the output of the achieved loop (i.e. the loop in which the controller obtained from the estimated model is applied to the true system). The results were all based on the transfer function variance formulas of [7], derived under the assumption that the model order tends to infinity, and it was observed in recent years that the use of these formulas for finite order models can sometimes lead to erroneous conclusions. This observation triggered a revival of interest in optimal design formulations based on *variance expressions for finite order models.*

Experiment design for robust control

Robust stability and robust performance criteria are typically expressed as constraints on frequency weighted expressions of the variance of the transfer function error, rather than as L_2 performance criteria. For example, a robust stability constraint is typically formulated as

$$Var\ G(e^{j\omega},\hat{\theta}_N) \leq W^{-1}(e^{j\omega})\ \forall\omega \tag{5}$$

where $W(e^{j\omega})$ is a frequency weighting function that takes account of closed-loop properties (e.g. robust stability condition). In order to formulate optimal input design problems in terms of control-oriented quality measures on $G(e^{j\omega},\hat{\theta}_N)$ such as in (5), using the finite model order formula (3) rather than

the asymptotic (in model order) variance formulas, several approaches can be taken.

One commonly used approach to go from parameter covariance to transfer function covariance is to use the following first order Taylor series approximation:

$$Var\ G(e^{j\omega}, \hat{\theta}_N) \approx \frac{\sigma_e^2}{N} \frac{\partial G^*(e^{j\omega}, \theta_0)}{\partial \theta} P_\theta \frac{\partial G(e^{j\omega}, \theta_0)}{\partial \theta} \tag{6}$$

This approach was adopted in [12] where it is shown that several useful H_∞ design criteria can be reformulated as weighted trace optimal input design problems subject to LMI constraints. A sensible open-loop optimal input design problem can then be formulated as follows:

$$\min_{\Phi_u(\omega)} tr[W(\theta_0)P_\theta] \quad \text{subject to} \quad \int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \leq \alpha, \quad \text{and } \Phi_u(\omega) \geq 0\ \forall\omega, \tag{7}$$

where α is some positive constant. This is still a difficult, infinite dimensional optimization problem. However, by the use of Schur complement, the problem can be reformulated as a convex optimization problem under Linear Matrix Inequality (LMI) constraints. The numerical solution of such problems became possible in the nineties with the advent of interior point optimization methods [13, 14]. The problem becomes finite dimensional if the input spectrum $\Phi_u(\omega)$ can be finitely parametrized. There are various ways of doing this, as noted earlier.

Another approach to optimal input design for robust control is based on the use of the ellipsoidal uncertainty set U_θ centred on $\hat{\theta}_N$:

$$U_\theta = \{\theta | (\theta - \hat{\theta}_N)^T P_\theta^{-1} (\theta - \hat{\theta}_N) < \chi^2\}. \tag{8}$$

It follows from the property (2) that the true parameter vector $\theta_0 \in \mathbb{R}^d$ belongs to U_θ with probability $\alpha(d, \chi^2) = Pr(\chi^2(d) \leq \chi^2)$, where $\chi^2(d)$ denotes the χ^2 distribution with d degrees of freedom. The results in [15, 16], which connect robust stability and robust performance measures directly to the ellipsoidal uncertainty region U_θ, allow one to formulate experiment design problems for robust control in terms of the minimization of some appropriate function of U_θ (or of P_θ) without the need for the intermediate step of transfer function variance estimation, which typically requires both a Taylor series approximation and/or a conservative step of overbounding of the uncertainty set.

The first open-loop optimal input design problem for robust control based on the direct use of the uncertainty ellipsoid U_θ was formulated in [5]. The robust stability measure minimized in that paper, with respect to the input spectrum $\Phi_u(\omega)$, was the worst-case ν-gap $\delta_{WC}(G(z, \hat{\theta}_N), \mathcal{D})$ between the identified model $G(z, \hat{\theta}_N)$ and all models in the Prediction Error uncertainty set $\mathcal{D} \triangleq \{G(z, \theta) | \theta \in U_\theta\}$:

$$\delta_{WC}(G(z, \hat{\theta}_N), \mathcal{D}) = \sup_{\theta \in U_\theta} \delta_\nu(G(z, \hat{\theta}_N), G(z, \theta)) \tag{9}$$

where the ν-gap is defined in [17]. One of the merits of the worst-case ν-gap is that it is directly related to the size of the set of its stabilizing controllers: the smaller the worst-case ν-gap of the uncertainty set $\mathcal{D}$, the larger is the set of controllers that stabilize all models in $\mathcal{D}$. The optimal input design problem solved in [5] was

$$\min_{\Phi_u} \delta_{WC}(G(z,\hat{\theta}_N),\mathcal{D}) \text{ subject to } \int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \leq \alpha, \quad \text{and } \Phi_u(\omega) \geq 0 \ \forall\omega. \tag{10}$$

The solution proposed in [5] is based on Tchebycheff system theory (see above); the optimal solution can always be obtained as a multisine.

Why do more work than is needed ?
The traditional approach to optimal input design, as exemplified by the problem formulations (7) or (10), has been to optimize some measure of the resulting uncertainty, subject to a constraint on the input signal power. However, in an identification for robust control setting, one should not spend more effort on the identification than what is needed for the design of a robust controller, under the constraint that this controller must achieve stability and a prespecified level of performance with all models in the uncertainty set. This idea has led to the recent concept of "least costly identification for control", which was first proposed in [18]. Instead of minimizing some measure of the uncertainty set, the objective is to deliver an uncertainty set that is just within the bounds required by the robust control specifications, and to do so at the smallest possible cost. In [19] open-loop identification is considered and the cost is then defined as the total input signal power. The idea of least costly (or minimum energy) identification experiment for control has been further developed in an open-loop framework in [20].

Here we present the formulation of this optimal experiment design objective in a closed-loop disturbance rejection setup, where no reference excitation is applied in normal operation[6]. The identification cost is then defined as the additional penalty that occurs in the control performance cost when an excitation signal is added for the purposes of doing the identification. The full technical details can be found in [21]. We first recall some basic results and tools about Prediction Error identification for closed-loop systems.

2 Prediction Error Identification Aspects

We consider the identification of the closed-loop system (1) using the full order model structure $\mathcal{M} = \{G(z,\theta),\ H(z,\theta)\},\ \theta \in \mathbf{R}^k$. In addition, we

[6] This setup is very reminiscent of M. Gevers' first practical experience with experiment design in a closed-loop disturbance rejection framework on the lakes of Canberra, Australia in 1985, when he and Keith Glover had jointly bought their first windsurfer and were identifying its dynamics.

assume throughout the paper that the model structure is globally identifiable at θ_0 [22]. This means that θ_0 is the only value of θ for which $G(z,\theta_0)$ and $H(z,\theta_0)$ represent the true system. Conditions for global identifiability of the commonly used model structures (ARX, ARMAX, BJ, etc) can be found in Theorem 4.2 of [22]. Note that they do not strictly require that all polynomial orders be known exactly.

This true system is assumed to be operated in closed loop with a controller C_{id} :

$$\mathcal{C} : u(t) = r(t) - C_{id}(z)y(t) \tag{11}$$

to be replaced by a better controller: see Fig. 1. In normal operation, the external excitation signal $r(t)$ is assumed to be zero. The objective is to perform a closed-loop identification experiment in order to estimate a model that must lead to a new robust controller with better performance. This is typically achieved by applying an external excitation signal r, even though we shall see that this is not always necessary. The closed-loop system can be written as:

$$\begin{aligned} y(t) &= S_{id}v(t) + \overbrace{G_0 S_{id} r(t)}^{y_r(t)} \\ u(t) &= -C_{id}S_{id}v(t) + \overbrace{S_{id}r(t)}^{u_r(t)} \end{aligned} \tag{12}$$

with $S_{id} = 1/(1+C_{id}G_0)$. Under normal operation, only the first terms (driven by v) appear in the input and output signals. During an identification experiment with an external excitation signal r, the controlled output and input of the closed-loop system contain added contributions y_r and u_r due to the excitation signal $r(t)$; these are perturbations with respect to the normal operating conditions.

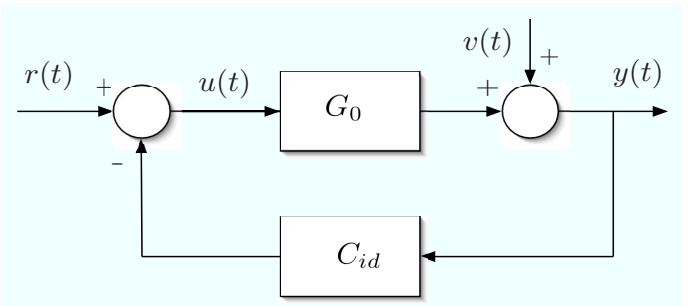

Fig. 1. Block-diagram of the closed-loop system

Consider now that a model $\hat{G}(z) = G(z,\hat{\theta}_N), \hat{H}(z) = H(z,\hat{\theta}_N)$ of the true system is identified using a direct Prediction Error identification method on the basis of N input-output data collected on the actual closed loop system with the controller C_{id} in the loop. If an external excitation signal $r(t)$ is

applied during this identification experiment, we assume that it is a sample taken from a quasi-stationary signal (see [22]) for which a power spectrum $\Phi_r(\omega)$ exists. The parameter vector estimate $\hat{\theta}_N$ is defined by:

$$\hat{\theta}_N \stackrel{\Delta}{=} \arg \min_{\theta} \frac{1}{N} \sum_{t=1}^{N} \epsilon^2(t, \theta) \tag{13}$$

where $\epsilon(t, \theta) \stackrel{\Delta}{=} H(z, \theta)^{-1} (y(t) - G(z, \theta)u(t))$. Note that $\epsilon(t, \theta)$ depends on the chosen signal $r(t)$ via (12).

We introduce the following cost function:

$$\bar{V}(\theta) = \bar{E}\epsilon^2(t, \theta) \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E\epsilon^2(t, \theta). \tag{14}$$

As shown in [22], if the identification experiment is "informative enough", then $\hat{\theta}_N$ tends w.p.1 to a minimum of the cost function $\bar{V}(\theta)$; by our standing assumption that the system is in the model set and that the model structure is globally identifiable at θ_0, this cost function has θ_0 as its unique minimum. The easiest and most common way to make a closed-loop experiment informative enough is to choose a reference signal that is persistently exciting of sufficient order (in closed-loop identification, the required order of the excitation is related to the degree of the sensitivity function S_{id}); however, a closed-loop experiment can also be informative enough even without any external excitation, provided the controller C_{id} is sufficiently complex, i.e. provided its degree is sufficient: see [21, 23].

In this paper, we shall assume throughout that the experimental conditions are informative enough. Thus we make the following Standing Assumption.

Assumption 2.1 *With reference to the closed-loop identification experiment presented above (see (12)-(13)), assume that the system is in the model set, that the model structure is globally identifiable at θ_0, and that the experimental conditions are informative enough so that the true parameter vector θ_0 is the only global minimum of the cost function $\bar{V}(\theta)$ of (14).*

We observe that this assumption is restrictive only in its requirements on the model structure. The requirements on the experimental conditions can always be satisfied. Assumption 2.1 ensures that the uncertainty region constructed around $\hat{\theta}_N$ contains θ_0 (and not another minimum of $\bar{V}(\theta)$) and that this uncertainty region is not infinitely large.

When Assumption 2.1 holds, we have the following result.

Lemma 2.1 *Consider the closed-loop identification experiment described above (see (12)-(13)) together with Assumption 2.1. Then the identified parameter vector $\hat{\theta}_N$ is asymptotically normally distributed around the true parameter vector θ_0, i.e. $\hat{\theta}_N - \theta_0$ converges in distribution to $\mathcal{N}(0, P_\theta)$, and the covariance matrix P_θ has the following expression [22]:*

$$P_\theta = \frac{\sigma_e^2}{N}\left(\bar{E}\left(\psi(t,\theta_0)\psi(t,\theta_0)^T\right)\right)^{-1} \tag{15}$$

with $\psi(t,\theta) = -\frac{\partial \epsilon(t,\theta)}{\partial \theta}$.

The covariance matrix P_θ depends on the experimental conditions (i.e. the data length N and the spectrum $\Phi_r(\omega)$ used during the identification), as evidenced by the following expression of P_θ^{-1}, which is easily deduced from (12) and (15):

$$P_\theta^{-1} = N \overbrace{\left(\frac{1}{\sigma_e^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_r(e^{j\omega},\theta_0)F_r(e^{j\omega},\theta_0)^*\Phi_r(\omega)d\omega\right)}^{\mathcal{P}_r^{-1}(\Phi_r(\omega),\theta_0,\sigma_e^2)}$$
$$+ N \overbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} F_e(e^{j\omega},\theta_0)F_e(e^{j\omega},\theta_0)^*d\omega\right)}^{\mathcal{P}_v^{-1}(\theta_0)} \tag{16}$$

Here, $F_r(z,\theta_0) = S_{id}\frac{\Lambda_G(z,\theta_0)}{H(z,\theta_0)}$, $F_e(z,\theta_0) = \frac{\Lambda_H(z,\theta_0)}{H(z,\theta_0)} - C_{id}S_{id}\Lambda_G(z,\theta_0)$, $\Lambda_G(z,\theta) = \frac{\partial G(z,\theta)}{\partial \theta}$ and $\Lambda_H(z,\theta) = \frac{\partial H(z,\theta)}{\partial \theta}$. Note that P_θ^{-1} is made up of a part depending on $\Phi_r(\omega)$ and a part which does not depend on $\Phi_r(\omega)$. Both parts are linear in N and both depend on the controller C_{id}. Note also that, in the vector F_r, the entries corresponding to the parameters that are only present in $H(z,\theta)$ are identically zero. The vector F_e has no identically zero entries; F_e is made up of the sum of two components: one component pertains to the parameters in $H(z,\theta)$ and one pertains to the parameters in $G(z,\theta)$.

Lemma 2.2 ([22]) *Consider the closed-loop identification experiment described above (see (12)-(13)) and assume that Assumption 2.1 holds. Then* P_θ^{-1} *is strictly positive definite:* $P_\theta^{-1} \succ 0$.

Using Lemmas 2.1 and 2.2, it is possible to define an uncertainty region $\mathcal{D}(\hat{\theta}_N, P_\theta)$ around the identified model which contains the unknown true system $G(z,\theta_0)$ at any desired probability level β [15, 24]:

$$\mathcal{D}(\hat{\theta}_N, P_\theta) = \left\{ G(z,\theta) = \frac{Z_N(z)\theta}{1+Z_D(z)\theta} | \theta \in U = \{\theta | (\theta-\hat{\theta}_N)^T P_\theta^{-1}(\theta-\hat{\theta}_N) < \chi\} \right\} \tag{17}$$

where χ is a real constant dependent on the chosen probability level β and Z_N, Z_D are row vectors containing powers of z^{-1} and zeros. The size of the uncertainty region $\mathcal{D}(\hat{\theta}_N, P_\theta)$ is a function of the covariance matrix P_θ and thus, by (16), a function of the design parameters N and $\Phi_r(\omega)$ used during the identification experiment.

3 Control Design Objectives and Control Design Method

As stated before, our aim is to replace the present controller C_{id} in the loop of Fig. 1 by a new controller $\hat{C}(z) = C(G(z, \hat{\theta}_N))$ that has better performance. We adopt the following control performance measure for a stable closed-loop system $[C\ G]$:

$$J(G, C, W_l, W_r) = \sup_{\omega} \bar{J}(\omega, G, C, W_l, W_r) \tag{18}$$

where

$$\bar{J}(\omega, G, C, W_l, W_r) = \sigma_{max}(W_l(e^{j\omega})F(G(e^{j\omega}), C(e^{j\omega}))W_r(e^{j\omega})) \tag{19}$$

$$F(G, C) \stackrel{\Delta}{=} \begin{pmatrix} \frac{GC}{1+GC} & \frac{G}{1+GC} \\ \frac{C}{1+GC} & \frac{1}{1+GC} \end{pmatrix}$$

Here $\sigma_{max}(A)$ denotes the largest singular value of A, and $W_l(z)$, $W_r(z)$ are given diagonal filters which reflect the performance specifications that we want to achieve with the true system. The performance measure (18) is quite general: $J(G, C, W_l, W_r) \leq 1$ ensures that the four entries of $W_l(z)F(G, C)W_r(z)$ have an H_∞ norm smaller than one. Simpler H_∞ criteria can be chosen as special cases. A controller C will be deemed *satisfactory* for the system G_0 if $[C\ G_0]$ is stable and if $J(G_0, C, W_l, W_r) \leq 1$.

As mentioned in the introduction, we want to design the new controller $\hat{C} = C(G(z, \hat{\theta}_N))$ using an identified model $\hat{G} = G(z, \hat{\theta}_N)$ of G_0. For this purpose, we use a pre-selected nominal control design method.

Assumption 3.1 *We have pre-selected a fixed nominal control design method which maps the identified model $G(z, \hat{\theta}_N)$ to one controller $C(G(z, \hat{\theta}_N))$ which stabilizes $G(z, \hat{\theta}_N)$ and achieves with this model a nominal performance level*

$$J(G(z, \hat{\theta}_N), C(G(z, \hat{\theta}_N)), W_l(z), W_r(z)) \leq \gamma < 1, \tag{20}$$

where γ is a fixed scalar, strictly smaller than 1.

One possible control design choice that satisfies Assumption 3.1 is to choose for $C(G(z, \hat{\theta}_N))$ the central controller of the four-block H_∞ control design method with performance objective (20).

If Assumption 3.1 holds, then the controller $\hat{C} = C(G(z, \hat{\theta}_N))$ designed from an identified model $\hat{G} = G(z, \hat{\theta}_N)$ will achieve $J(\hat{G}, \hat{C}, W_l, W_r) \leq \gamma < 1$. When this controller $\hat{C}$ is applied to the true system G_0, the achieved performance will generically be poorer than the designed performance. However, by choosing the design criterion (20) with $\gamma < 1$, we ensure that there is a whole set of systems $G(z)$ around $G(z, \hat{\theta}_N)$ that are also stabilized by $\hat{C}$ and that achieve $J(G, \hat{C}, W_l, W_r) \leq 1$. In the sequel, we will denote by $\mathcal{D}_{adm}(\hat{\theta}_N)$ the largest set of systems $G(z)$ having these properties.

4 Demands on the Identification Experiment

Our objective is to determine the experimental conditions (N and $\Phi_r(\omega)$) of the identification experiment on the loop $[C_{id}\ G_0]$ in such a way that the model $\hat{G}$, identified through this experiment, delivers a controller $\hat{C}$ which stabilizes the unknown G_0 and achieves $J(G_0, \hat{C}, W_l, W_r) \leq 1$. Since G_0 is unknown but lies (with probability β) in the uncertainty region $\mathcal{D}(\hat{\theta}_N, P_\theta)$ identified along with $\hat{G}$, this performance constraint will be replaced by the following checkable constraint[7].

Constraint 4.1 *The experimental conditions (N and $\Phi_r(\omega)$) of the identification experiment on the loop $[C_{id}\ G_0]$ (see Section 2) must be such that the identified model $\hat{G} = G(z, \hat{\theta}_N)$ and the identified uncertainty region $\mathcal{D}(\hat{\theta}_N, P_\theta)$ have the property that $J(G, \hat{C}, W_l, W_r) \leq 1$ for all $G(z) \in \mathcal{D}(\hat{\theta}_N, P_\theta)$, where $\hat{C}$ is the controller designed from $\hat{G}$ using the control design method presented in Assumption 3.1.*

Since $J(G, \hat{C}, W_l, W_r) \leq 1$ for all $G(z)$ in the set $\mathcal{D}_{adm}(\hat{\theta}_N)$ defined in the last paragraph of Section 3, Constraint 4.1 imposes that $\mathcal{D}(\hat{\theta}_N, P_\theta)$ is a subset of this set $\mathcal{D}_{adm}(\hat{\theta}_N)$. We now discuss the requirements on the identification design that will guarantee this.

Remember that $\mathcal{D}_{adm}(\hat{\theta}_N)$ is the largest set of systems $G(z)$ around $G(z, \hat{\theta}_N)$ for which $J(G, \hat{C}, W_l, W_r) \leq 1$. By Assumption 2.1 and Lemma 2.2, $\mathcal{D}(\hat{\theta}_N, P_\theta) \subseteq \mathcal{D}_{adm}(\hat{\theta}_N)$ is always achievable if N and/or $\Phi_r(\omega)$ are chosen large enough: see (16). Moreover, we have the following trade-offs: the larger N is chosen, the *smaller* $\Phi_r(\omega)$ can be while still verifying Constraint 4.1; conversely, the *larger* $\Phi_r(\omega)$ is chosen, the smaller N can be while still verifying this constraint.

The sets $\mathcal{D}_{adm}(\hat{\theta}_N)$ and $\mathcal{D}(\hat{\theta}_N, P_\theta)$ are both a function of the parameter vector $\hat{\theta}_N$ that we want to identify. Moreover, besides being a function of $\Phi_r(\omega)$ and N, P_θ is also a function of the unknown quantities θ_0 and σ_e^2. Consequently, whatever the method we use to determine experimental conditions $\Phi_r(\omega)$ and N satisfying Constraint 4.1, it will need to be based on some initial estimates $\theta_{o,est}$, $\hat{\theta}_{N,est}$ and $\sigma^2_{e,est}$ of those unknown quantities. In this context, we will state that Constraint 4.1 is satisfied for given N and $\Phi_r(\omega)$ when $\mathcal{D}(\hat{\theta}_{N,est}, P_{\theta,est}) \subseteq \mathcal{D}_{adm}(\hat{\theta}_{N,est})$ where $P_{\theta,est}$ is computed using (16) with θ_0 and σ_e^2 replaced by $\theta_{o,est}$ and $\sigma^2_{e,est}$. Note that, in the methods we present in the sequel, we will not determine the set $\mathcal{D}_{adm}$ explicitly, but implicitly (see Theorems 7.1 and 7.2).

[7] If $J(G, \hat{C}, W_l, W_r) \leq 1$ with all $G(z) \in \mathcal{D}(\hat{\theta}_N, P_\theta)$, then, under mild assumptions, $\hat{C}(z) = C(G(z, \hat{\theta}_N))$ also stabilizes all $G \in \mathcal{D}(\hat{\theta}_N, P_\theta)$.

5 The Cost of a Closed-Loop Identification Experiment

It is clear that many possible choices of experimental conditions allow one to fulfill Constraint 4.1. Among those, we seek to determine an identification experiment with the smallest possible cost. In this section, we give a precise definition of the cost of an identification experiment in the context where the closed-loop system operates with an acting controller C_{id} and with a disturbance rejection performance objective.

As mentioned in Section 2, in normal operation the signals $u(t)$ and $y(t)$ are given by:

$$y(t) = S_{id}v(t), \qquad u(t) = -C_{id}S_{id}v(t). \tag{21}$$

By applying an external signal $r(t)$ to the loop during the identification, we introduce *additional disturbances* $y_r(t)$ and $u_r(t)$ on top of the normal operation signals: see (12). Those disturbances represent the cost of the identification experiment, since they entail a performance degradation. The ideal closed-loop identification experiment would be one in which the normal operation signals $u(t)$ and $y(t)$ are used for a certain length N without any external excitation, i.e. with $r(t) = 0$. We show in Section 6 that such costless identification experiment can, in certain circumstances, lead to fulfillment of the Constraint 4.1. We also show how to compute the minimum number N_{min} of measurements that are necessary to reach this objective.

In the cases where Constraint 4.1 can not be achieved with $r(t) = 0$, the application of a nonzero external signal $r(t)$ for a certain amount of time is unavoidable, but we show how N and $\Phi_r(\omega)$ can be chosen in order to achieve Constraint 4.1 with minimal cost. This cost can be a function of either the experiment time N, the power of the perturbations y_r and u_r, or a combination of both. In the sequel, we consider three different situations which are representative of practical situations, and determine for each of them how we can optimally choose the experimental conditions.

Situation 1. The cost of the identification is mainly determined by the duration N of the identification experiment. Based on the trade-off discussed in Section 4 between excitation power and duration of the experiment, the power spectrum $\Phi_r(\omega)$ of the to-be-applied signal $r(t)$ is in this case chosen at each frequency as large as the constraints on the actuators allow. For such fixed $\Phi_r(\omega)$, the optimal experiment time can subsequently be determined via an optimization problem yielding the smallest identification time N_{min} satisfying Constraint 4.1.

Situation 2. Situation 2 is the converse situation: the cost of the identification is mainly determined by the power of the perturbations $y_r(t)$ and $u_r(t)$ due to the excitation signal $r(t)$. Based on the same trade-off, the experiment time N is in this case chosen as large as is allowed. For such fixed N, the optimal power spectrum $\Phi_r(\omega)$ can then be determined via an optimization problem

whose objective is to minimize the following cost function $\mathcal{J}_r$ representing the total disturbance power, subject to satisfaction of the Constraint 4.1:

$$\begin{aligned}\mathcal{J}_r &= \alpha_y \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{y_r}(\omega)\, d\omega \right) + \alpha_u \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{u_r}(\omega)\, d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\alpha_y |G_0(e^{j\omega}) S_{id}(e^{j\omega})|^2 + \alpha_u |S_{id}(e^{j\omega})|^2 \right) \Phi_r(\omega)\, d\omega \end{aligned} \tag{22}$$

where α_y and α_u are scalars chosen by the designer to reflect the relative importance of the costs due to each of the perturbation signals, and where $\Phi_{y_r}(\omega)$ and $\Phi_{u_r}(\omega)$ are the power spectra of these disturbance signals $y_r(t)$ and $u_r(t)$.

Situation 3. Situation 3 is the situation where N and $\Phi_r(\omega)$ are both important in the cost of the identification. In this situation, we can determine the optimal spectrum $\Phi_r(\omega)$ such as in Situation 2 for different values of the length N. Since, for increasing values of N, the optimal cost function $\mathcal{J}_r$ decreases, such approach allows one to find the "optimal" combination for the duration of the identification experiment and the induced disturbance on the input and output signals.

6 Identification Experiments Without External Excitation

In this section we examine the situation where Constraint 4.1 can be achieved using an identification experiment on the closed loop $[C_{id}\ G_0]$ without any external excitation signal $r(t)$, i.e. using only the excitation due to the noise $v(t)$ (see (12) with $r = 0$). We have the following result.

Theorem 6.1 *Consider a closed-loop identification experiment as presented in Section 2 with $r(t) = 0$ and assume that Assumption 2.1 holds. Then, Constraint 4.1 can always be verified by using for the identification a set of input-output data (21) of sufficient length N.*

Proof. When $r = 0$, expression (16) of P_θ^{-1} becomes:

$$P_\theta^{-1} = N \mathcal{P}_v^{-1}(\theta_0). \tag{23}$$

By Assumption 2.1 and Lemma 2.2, we have $\mathcal{P}_v^{-1}(\theta_0) \succ 0$. Therefore, with N sufficiently large, $P_\theta^{-1} = N\mathcal{P}_v^{-1}(\theta_0)$ can be made such that $\mathcal{D}(\hat{\theta}_N, P_\theta) \subseteq \mathcal{D}_{adm}(\hat{\theta}_N)$ for any set $\mathcal{D}_{adm}(\hat{\theta}_N)$ around $G(z, \hat{\theta}_N)$, which implies that Constraint 4.1 holds. ∎

Theorem 6.1 shows that, if Assumption 2.1 holds with $r = 0$, the identification leading to a new and satisfactory controller $\hat{C}$ for G_0 can be achieved without applying any external excitation: we just need to measure the input

and output signal in normal operation for a sufficient amount of time. For this theorem to be of practical use, we need to examine under what conditions Assumption 2.1 holds when $r = 0$, and which smallest data length N_{min} is required for the robustness Constraint 4.1 to be satisfied when $r = 0$.

The computation of N_{min} for the case $r = 0$ is a special case of the computation of N_{min} for a given reference excitation spectrum $\Phi_r(\omega)$; it will be presented in the next section. As for the first question, a necessary and sufficient condition for Assumption 2.1 to hold is that $\epsilon(t, \theta) = \epsilon(t, \theta_0) \Longrightarrow \theta = \theta_0$ [23]. In the case where $r = 0$, this condition specializes to the following result.

Lemma 6.1 ([23]) *Consider the closed-loop identification configuration of Section 2 with $r = 0$. Then, Assumption 2.1 holds if and only if, for any θ,*

$$H^{-1}(z, \theta)(1 + C_{id}G(z, \theta)) = H^{-1}(z, \theta_0)(1 + C_{id}G(z, \theta_0)) \Longrightarrow \theta = \theta_0. \quad (24)$$

A necessary condition for this identifiability condition to hold, in the case considered here of a linear time-invariant regulator C_{id}, is that the regulator be sufficiently complex (i.e. of sufficiently high order): see Section 13.4 of [22]. One can make this statement more precise by considering specific model structures. This has been done in [23] (see Complement C10.1) for the case of an ARMAX or ARX model structure, and in [21] for the case where an OE or BJ model structure is used. The conditions are essentially degree constraints on the structure of the controller C_{id}.

7 Least Costly Identification Experiments for Control

An identification experiment without external excitation may be impossible for two reasons: i) the initial controller C_{id}, which is often not chosen by the user, is of lower complexity than required, or ii) the data length required to satisfy Constraint 4.1 (see Theorem 6.1) is unrealistic (e.g. N_{min} corresponds to one year of data). We now address the problem of computing the least costly identification experiment for control, as has been defined by the problems presented at the end of Section 4. Those problems involve the computation under Constraint 4.1 of either the smallest data length for a given $\Phi_r(\omega)$ (Situation 1) or of the power spectrum $\Phi_r(\omega)$ minimizing $\mathcal{J}_r$ for a given N (Situations 2 and 3).

7.1 Shortest Identification Experiment for Control with Fixed $\Phi_r(\omega)$

The first experiment design problem can be formulated as follows.

Experiment Design Problem 1. *Consider the closed-loop identification experiment of Section 2. Consider also that the power spectrum $\Phi_r(\omega)$ of the*

excitation signal $r(t)$ is given. Determine then the smallest length N of an excitation signal $r(t)$ with power spectrum $\Phi_r(\omega)$ that must be applied to $[C_{id}\ G_0]$ in order to fulfill Constraint 4.1.

We show that this problem can be expressed as an LMI-based optimization problem [14]. For this purpose, we first express the robust performance constraint $\bar{J}(\omega, G, \hat{C}, W_l, W_r) \leq 1\ \forall G \in \mathcal{D}(\hat{\theta}_N, P_\theta)$ at one particular frequency ω as an LMI, linear in P_θ^{-1}. Note that, according to (18), $J(G, \hat{C}, W_l, W_r) \leq 1$ $\forall G \in \mathcal{D}(\hat{\theta}_N, P_\theta) \iff \bar{J}(\omega, G, \hat{C}, W_l, W_r) \leq 1\ \forall \omega$ and $\forall G \in \mathcal{D}(\hat{\theta}_N, P_\theta)$.

Proposition 2 *Consider the controller $\hat{C} = C(G(z, \hat{\theta}_N))$ designed from the model $G(z, \hat{\theta}_N)$ using the control design method presented in Assumption 3.1. Consider also the set $\mathcal{D}(\hat{\theta}_N, P_\theta)$ defined in (17). Then $\hat{C}$ achieves $\bar{J}(\omega, G, \hat{C}, W_l, W_r) \leq 1$ with all G in $\mathcal{D}(\hat{\theta}_N, P_\theta)$ if and only if $\exists\ \tau(\omega) > 0$ ($\tau(\omega) \in \mathbf{R}$) and a skew-symmetric matrix $\mathcal{L}(\omega) \in \mathbf{R}^{(k+1)\times(k+1)}$ (i.e. $\mathcal{L}(\omega) = -\mathcal{L}(\omega)^T$) such that*

$$\tau(\omega) E(\omega, \hat{\theta}_N) - \mathcal{R}(\hat{\theta}_N) + j\ \mathcal{L}(\omega) \leq 0 \tag{25}$$

with $j = \sqrt{-1}$ and

$$\mathcal{R}(\hat{\theta}_N) = \begin{pmatrix} I_k \\ -\hat{\theta}_N^T \end{pmatrix} P_\theta^{-1} \begin{pmatrix} I_k \\ -\hat{\theta}_N^T \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & -\chi \end{pmatrix}$$

$$E(\omega, \hat{\theta}_N) = \Omega^*(e^{j\omega}) \begin{pmatrix} I_4 & 0 \\ 0 & -1 \end{pmatrix} \Omega(e^{j\omega})$$

$$\Omega(z) = \left(\begin{array}{c|c} \left(I_2 \otimes \left(W_r \begin{pmatrix} \hat{C} \\ 1 \end{pmatrix}\right)\right) W_l & 0 \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} Z_N & 0 \\ Z_D & 1 \\ Z_D + \hat{C} Z_N & 1 \end{pmatrix}$$

The symbol $\otimes$ denotes the Kronecker product.

Proof. See [21]. ■

The previous proposition shows that Constraint 4.1 can be replaced by LMI's at each frequency, linear in P_θ^{-1}. Expression (16) shows that P_θ^{-1} is linear in the decision variable N of Experiment Design Problem 1. By combining these two facts, it is easy to see that Experiment Design Problem 1 would be solvable exactly if P_θ^{-1} was not a function of θ_0 and σ_e^2, and if condition (25) was not a function of the to-be-identified $\hat{\theta}_N$. This difficulty is inherent to all experiment design problems [22] and is generally circumvented by using a-priori estimates for those quantities: $\theta_{o,est}$, $\sigma_{e,est}^2$ and $\hat{\theta}_{N,est}$. The problem can then be solved using the LMI optimization problem of Theorem 7.1 below. Note that $\hat{\theta}_{N,est}$ will often be chosen equal to $\theta_{o,est}$.

Theorem 7.1 *Consider the approximations $\theta_0 \approx \theta_{o,est}$, $\hat{\theta}_N \approx \hat{\theta}_{N,est}$ and $\sigma_e^2 \approx \sigma_{e,est}^2$ and the shorthand notations: $\mathcal{P}_r^{-1}(\Phi_r(\omega)) = \mathcal{P}_r^{-1}(\Phi_r(\omega), \theta_{o,est}, \sigma_{e,est}^2)$,*

$\mathcal{P}_v^{-1} = \mathcal{P}_v^{-1}(\theta_{o,est})$ *(see (16)) and* $E(\omega) = E(\omega, \hat{\theta}_{N,est})$ *(see (25)). Then, the minimum duration* N *which solves Experiment Design Problem 1 is the solution (rounded up to the nearest integer) of the following LMI optimization problem:*

$$\min N$$

under the constraint that $\exists$ *a frequency function* $\tau(\omega)$ *valued in* $\mathbf{R}$ *and a frequency-dependent skew-symmetric matrix* $\mathcal{L}(\omega)$ *valued in* $\mathbf{R}^{(k+1)\times(k+1)}$ *such that*

$$\tau(\omega)E(\omega) - \begin{pmatrix} I_k \\ -\hat{\theta}_{N,est}^T \end{pmatrix} \left(N\ (\mathcal{P}_r^{-1}(\Phi_r(\omega)) + \mathcal{P}_v^{-1})\right) \begin{pmatrix} I_k \\ -\hat{\theta}_{N,est}^T \end{pmatrix}^T - \begin{pmatrix} 0 & 0 \\ 0 & -\chi \end{pmatrix} + j\ \mathcal{L}(\omega) \leq 0 \quad \forall\omega \tag{26}$$

Proof. Direct consequence of Proposition 2 and the expression of P_θ^{-1} in (16). ■

Comment 1. Condition (26) must be considered at every frequency. This is impossible in practice. The optimal N can nevertheless be approximated by using a finite frequency grid. An exact but more cumbersome solution consists of using the Kalman-Yakubovitch-Popov (KYP) lemma [25]; see [21] for details.

Comment 2. As stated in Section 6, the minimal data length for an identification experiment without external excitation can be determined via Theorem 7.1 as well, by setting $\Phi_r(\omega) = 0$ in (26).

7.2 Least costly identification experiment with fixed data length

We now examine the second situation presented at the end of Section 4, in which the data length N is fixed and one seeks to satisfy Constraint 4.1 with a power spetrum $\Phi_r(\omega)$ that minimizes the identification cost $\mathcal{J}_r$ defined in (22). We restrict our search to signals whose power spectrum $\Phi_r(\omega)$ can be written as [26]:

$$\Phi_r(\omega) = R_r(0) + 2\ \sum_{i=1}^{m} R_r(i)cos(i\omega)\ \geq 0 \quad \forall\omega \tag{27}$$

where m is a positive integer selected by the user. The parameters $R_r(i)$ $(i = 0...m)$ can be interpreted as the auto-correlation sequence of a signal that has been generated by a white noise passing through an FIR filter of length $m + 1$. If we select $R_r(i) = \sigma^2 \frac{N_c - i}{N_c}$ and $m = N_c$, then $\Phi_r(\omega)$ represents the power spectrum of a Random Binary Signal with clock period N_c and maximal

amplitude σ. Such a parametrization can be very useful if the amplitude of the time domain signal is constrained.

An important property of the parametrization (27) is that P_θ^{-1} and $\mathcal{J}_r$ (see (16) and (22)) are affine functions of the design variables $R_r(i)$ $(i = 0...m)$, as we show in the following two propositions. Note that other parametrizations of $\Phi_r(\omega)$ have the same property and could therefore also be considered here: e.g. $\Phi_r(\omega) = \sum_{i=1}^{m} R_r(i)\delta(\omega - \omega_i)$ corresponding to a multisine signal $r(t)$ [27], or $\Phi_r(\omega) = \sum_{i=0}^{m} R_r(i)\left(\mathcal{B}_i(e^{j\omega}) + \mathcal{B}_i^*(e^{j\omega})\right)$ where $\mathcal{B}_i(e^{j\omega})$ are preselected basis functions [28].

Proposition 3 *Consider the expression (16) of P_θ^{-1} and let $\Phi_r(\omega)$ be parametrized by (27). Let $\tilde{M}_k(\theta_0)$ be the sequence of Markov parameters of $F_r F_r^*$ i.e. $F_r(e^{j\omega}, \theta_0)F_r(e^{j\omega}, \theta_0)^* = \sum_{k=-\infty}^{\infty} \tilde{M}_k(\theta_0)e^{-jk\omega}$ with $F_r(z, \theta_0)$ as defined in (16). Then, $P_\theta^{-1} \in \mathbf{R}^{k\times k}$ can be written as:*

$$P_\theta^{-1} = \bar{M}(\theta_0) + \sum_{i=0}^{m} M_i(\theta_0, \sigma_e^2)\ R_r(i)$$

where $\bar{M}(\theta_0) = N\mathcal{P}_v^{-1}(\theta_0)$, $M_0(\theta_0, \sigma_e^2) = \frac{N}{\sigma_e^2}\ \tilde{M}_0(\theta_0)$, and $M_i(\theta_0, \sigma_e^2) = \frac{N}{\sigma_e^2}\ (\tilde{M}_i(\theta_0) + \tilde{M}_i^T(\theta_0))$ for $i = 1...m$.

Proof. Direct consequence of Result 5.6 of [26] applied to the closed-loop expression for P_θ^{-1} as given in (16). ■

Proposition 4 *Consider the cost function $\mathcal{J}_r$ defined in (22) and let $\Phi_r(\omega)$ be parametrized by (27). Then $\mathcal{J}_r$ can also be written as:*

$$\mathcal{J}_r = [\alpha_y c_0(\theta_0) + \alpha_u d_0(\theta_0)]R_r(0) + 2\sum_{i=1}^{m}[\alpha_y c_i(\theta_0) + \alpha_u d_i(\theta_0)]R_r(i),$$

where the coefficients $c_i(\theta_0)$ and $d_i(\theta_0)$ are the Markov parameters of $G_0 G_0^* S_{id} S_{id}^*$ and $S_{id} S_{id}^*$ respectively, i.e.

$$G_0(e^{j\omega})G_0(e^{j\omega})^* S_{id}(e^{j\omega})S_{id}(e^{j\omega})^* = \sum_{k=-\infty}^{\infty} c_k(\theta_0)e^{-jk\omega}$$

and $S_{id}(e^{j\omega})S_{id}(e^{j\omega})^* = \sum_{k=-\infty}^{\infty} d_k(\theta_0)e^{-jk\omega}$.

Proof. Direct consequence of Result 5.4 of [26] applied to $y_r(t) = G_0 S_{id} r(t)$ and $u_r(t) = S_{id} r(t)$. ■

With the parametrization (27) for $\Phi_r(\omega)$, the experiment design problem corresponding to Situation 2 can then be formulated as follows.

Experiment Design Problem 2: *Consider the closed-loop identification experiment of Section 2 with a fixed number N of data. Determine the parameters $R_r(i)$ $(i = 0...m)$ of the spectrum $\Phi_r(\omega)$ in (27) which minimize $\mathcal{J}_r$, subject to satisfaction of the Constraint 4.1.*

The experiment design problem described above would be solvable exactly, using the results of Propositions 3, 4 and 2, if the parametrizations of P_θ^{-1} and $\mathcal{J}_r$ with respect to the design variables $R_r(i)$ were not functions of the unknown θ_0 and σ_e^2, and if condition (25) was not a function of the to-be-identified $\hat{\theta}_N$. This difficulty is again circumvented by using a-priori estimates for those quantities: $\theta_{o,est}$, $\sigma^2_{e,est}$ and $\hat{\theta}_{N,est}$. The solution is then obtained by solving the LMI optimization problem described in Theorem 7.2 below.

Theorem 7.2 *Consider Experiment Design Problem 2. Consider also the approximations $\theta_0 \approx \theta_{o,est}$, $\hat{\theta}_N \approx \hat{\theta}_{N,est}$ and $\sigma_e^2 \approx \sigma^2_{e,est}$ and the shorthand notations: $c_i = c_i(\theta_{o,est})$, $d_i = d_i(\theta_{o,est})$, $\bar{M} = \bar{M}(\theta_{o,est})$, $M_i = M_i(\theta_{o,est}, \sigma^2_{e,est})$ and $E(\omega) = E(\omega, \hat{\theta}_{N,est})$. Then the auto-correlation sequence $R_r(i)$ $(i = 0...m)$ which solves Experiment Design Problem 2 is the solution of the following LMI optimization problem:*

$$\min_{R_r(i)(i=0...m)} [\alpha_y c_0 + \alpha_u d_0] R_r(0) + 2 \sum_{i=1}^{m} [\alpha_y c_i + \alpha_u d_i] R_r(i)$$

under the constraint that there exists a symmetric matrix Q of appropriate dimension, a frequency function $\tau(\omega)$ valued in $\mathbf{R}$ and a frequency-dependent skew-symmetric matrix $\mathcal{L}(\omega)$ valued in $\mathbf{R}^{(k+1)\times(k+1)}$ such that

$$\tau(\omega) E(e^{j\omega}) - \begin{pmatrix} I_k \\ -\hat{\theta}^T_{N,est} \end{pmatrix} \left(\bar{M} + \sum_{i=0}^{m} \; M_i \; R_r(i) \right) \begin{pmatrix} I_k \\ -\hat{\theta}^T_{N,est} \end{pmatrix}^T - \begin{pmatrix} 0 & 0 \\ 0 & -\chi \end{pmatrix} + j \; \mathcal{L}(\omega) \leq 0 \quad \forall \omega \tag{28}$$

$$\text{and that} \qquad \begin{pmatrix} Q - A^T Q A & C^T - A^T Q B \\ C - B^T Q A & D + D^T - B^T Q B \end{pmatrix} \geq 0 \tag{29}$$

with the following definitions of A, B, C, D:

$$A = \begin{pmatrix} 0 & 0 \\ I_{m-1} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & ... & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} R_r(1) & R_r(2) & ... & R_r(m) \end{pmatrix} \qquad D = \tfrac{R_r(0)}{2}$$

The optimal spectrum $\Phi_r(\omega)$ can thereafter be computed using (27).

Proof. The existence of a symmetric matrix Q such that (29) holds is a necessary and sufficient condition for $R_r(0) + 2 \sum_{i=1}^{m} R_r(i) cos(i\omega)$ to be positive at each ω and thus for (27) to represent a spectrum, as shown in [26] via the Positive Real Lemma. Consequently, the result in this theorem is a direct consequence of Propositions 3, 4 and 2. ■

Comment. Condition (28) must be considered at every frequency, which is impossible in practice. The optimal $\Phi_r(\omega)$ can nevertheless be approximated by using a finite frequency grid. An exact but more cumbersome solution consists of using the Kalman-Yakubovitch-Popov (KYP) lemma [25], as shown in [21].

7.3 Dealing with the Unknown Quantities θ_0, $\hat{\theta}_N$ and σ_e^2

Theorems 7.1 and 7.2 provide solutions to the optimal identification experiment design problems defined above. However, these solutions require that an approximation of θ_0, $\hat{\theta}_N$ and σ_e^2 be used. If those approximations are not accurate, this could lead to poor results. Here we present a procedure for dealing with those unknown variables.

Let us first recall the problem setup. One wants to replace the existing controller C_{id}, which operates on the true system, by a new controller via a new identification of the true system. In order to design the experimental conditions of this identification experiment, one needs reliable estimates of the unknown quantities θ_0, $\hat{\theta}_N$ and σ_e^2. It is very often the case that estimates for θ_0 and σ_e^2 are already available, because the initial controller C_{id} has typically been computed from an initial identified model. This initial identification typically delivers estimates not only of θ_0 and σ_e^2, but also of uncertainty regions for those quantities. Moreover, it is also possible to deduce from this initial identification a (truncated) Gaussian probability density function which defines the likelihood of each element of these uncertainty regions. If this is not possible, the density functions are then chosen uniform over the uncertainty regions. The estimate, the uncertainty region, and the probability density function of the to-be-identified $\hat{\theta}_N$ are typically chosen equal to those of θ_0.

To summarize, from the initial identification, one can assume that $q_0 = \left(\theta_0^T \ \hat{\theta}_N^T \ \sigma_e^2\right)^T$ lies in a set $\mathcal{Q}$ and that the likelihood of the event $q = q_0$ is given by a probability function $p(q)$. Based on this information, one can robustify the procedure that consists in adopting a unique and possibly poor estimate of q_0 for the design of the experimental conditions by adopting instead an approach based on randomized algorithms (see e.g. [29, 30]). We briefly describe such approach.

In the case of Experiment Design Problem 1, one wants to determine the smallest duration N for which the Constraint 4.1 is verified, whatever the value of q_0. For this specific problem, since one assumes that q_0 lies in $\mathcal{Q}$, this is equivalent to computing an estimate[8] $\hat{N}$ of $\sup_{q\in\mathcal{Q}} N_q$ where N_q is the solution obtained by Theorem 7.1 with the approximation q. Considering N_q as a function of q, this can be done [30] with accuracy ϵ and confidence[9] δ by generating $n \geq ln(\delta^{-1})/ln((1-\epsilon)^{-1})$ estimates q_j of q_0 according to the

[8] The exact computation is NP-hard.

[9] This means that $Pr(Pr(N_q > \hat{N}) \leq \epsilon) \geq 1 - \delta$.

probability density function $p(q)$, and by determining N_{q_j} for each of these estimates q_j using Theorem 7.1. The estimate $\hat{N}$ of $\sup_{q\in\mathcal{Q}} N_q$ is then given by $\sup_{q_j\ (j=1...n)} N_{q_j}$.

In the case of Experiment Design Problem 2, the approach above can not be considered since one determines the parameters $R_r(i)$ $(i = 0...m)$ of $\Phi_r(\omega)$ rather than $\Phi_r(\omega)$ itself. The so-called scenario-approach can then be considered (see e.g. [29, Chapter 12]). This approach is also based on a randomized algorithm which uses the probability density function $p(q)$. The main difference between the two approaches is that, for Experiment Design Problem 1, Theorem 7.1 is applied a fixed number of times for different estimates of q_0 that are randomly generated with $p(q)$; while for Experiment Design Problem 2, the optimization problem is solved only once but with several robust performance constraints (28), each of them evaluated at a different estimate of q_0 randomly generated with $p(q)$.[10]

8 Simulation Results

In order to illustrate our results, we consider the following ARX system [31] as the true system:

$$y(t) = \frac{z^{-3}B_0(z)}{A_0(z)}u(t) + \frac{1}{A_0(z)}e(t) \tag{30}$$

with $B_0(z) = 0.10276 + 0.18123z^{-1}$, $A_0(z) = 1 - 1.99185z^{-1} + 2.20265z^{-2} - 1.84083z^{-3} + 0.89413z^{-4}$, and $e(t)$ a realization of a white noise signal of variance $\sigma_e^2 = 0.5$.

The control performance criterion $J(G, C, W_l, W_r)$ focuses on the sensitivity function. It is defined as in (18) with the filters:

$$W_l(z) = diag\,(0, W(z))\ W_r(z) = diag\,(0, 1)\ W(z) = \tfrac{0.5165-0.4632z^{-1}}{1-0.999455z^{-1}}.$$

The true system initially operates in closed loop with a controller C_{id} which has been designed using an initial estimate of the true system $\theta_{o,est} = (-1.9755, 2.1965, -1.8495, 0.8881, 0.0817, 0.172)^T$ and the 4-block H_∞ control design method of [32] that satisfies Assumption 3.1:

$$\begin{aligned} C_{id} = {} & \frac{0.036249(z + 0.9244)(z^2 - 1.951z + 1.101)}{(z - 0.9995)(z^2 - 1.002z + 0.3641)} \\ & \times \frac{(z^2 - 0.5109z + 0.8248)(z^2 - 0.1828z + 0.9416)}{(z^2 - 1.279z + 0.835)(z^2 - 0.1746z + 0.9229)} \end{aligned} \tag{31}$$

[10] For each of these robust performance constraints (28), the frequency functions $\tau(\omega)$ and $\mathcal{L}(\omega)$ are different.

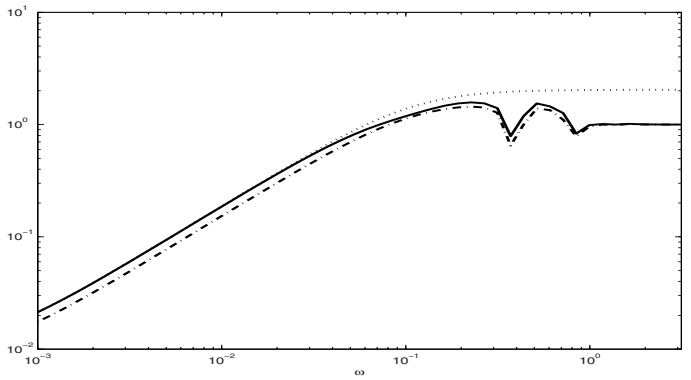

Fig. 2. Identification without external excitation, with $N = 4901$: $\sup_{G\in\mathcal{D}(\hat{\theta}_N,P_\theta)}\left|\frac{1}{1+\hat{C}(e^{j\omega})G(e^{j\omega})}\right|$ (solid), $|(1+\hat{C}\hat{G})^{-1}|$ (dashdot) and $|W|^{-1}$ (dotted)

Identification without external excitation. The complexity of the initial controller C_{id} is sufficient for Assumption 2.1 to hold with $r(t) = 0$. Thus, Constraint 4.1 can be verified with an identification experiment that uses noise excitation only (see 21) provided the experiment is of sufficient length. The minimal length required can then be determined using Theorem 7.1 with $\Phi_r(\omega) = 0\ \forall\omega$. This theorem is applied here using the approximations $\theta_0 \approx \theta_{o,est}$ and $\hat{\theta}_N \approx \theta_{o,est}$ (an estimate of σ_e^2 is not necessary since $\mathcal{P}_v^{-1}(\theta_0)$ is not a function of σ_e^2). This delivers a minimal length $N_{min} = 4901$.

In order to verify the validity of this result, we have measured 4901 samples of the signals $y(t)$ and $u(t)$ obtained in normal operation on the loop $[C_{id}\ G_0]$ and we have identified a model $\hat{G} = G(z, \hat{\theta}_N)$ along with its uncertainty region $\mathcal{D}(\hat{\theta}_N, P_\theta)$. From $\hat{G}$, we have then designed a controller $\hat{C}$ using the method of [32] and we have verified whether $\hat{C}$ achieves $J(G, \hat{C}, W_l, W_r) \leq 1$ with all G in $\mathcal{D}(\hat{\theta}_N, P_\theta)$, or equivalently $\left|\frac{1}{1+\hat{C}(e^{j\omega})G(e^{j\omega})}\right| \leq |W(e^{j\omega})|^{-1}$ for all G in $\mathcal{D}(\hat{\theta}_N, P_\theta)$. This is indeed the case as can be seen in Figure 2. Moreover, we also observe in Figure 2 that $\sup_{G\in\mathcal{D}(\hat{\theta}_N,P_\theta)}\left|\frac{1}{1+\hat{C}(e^{j\omega})G(e^{j\omega})}\right| = |W(e^{j\omega})|^{-1}$ in the low frequencies. Consequently, $N = 4901$ is indeed the smallest N for which Constraint 4.1 holds with $\Phi_r(\omega) = 0\ \forall\omega$.

Sensitivity to the initial estimates. In our example, the initial estimate $\theta_{o,est}$ chosen to approximate the unknown quantities θ_0 and $\hat{\theta}_N$ has delivered accurate results, as shown by Figure 2. This may not always be the case. Thus, it is safer to compute the minimal data length N using the method proposed in Section 7.3. We illustrate the application of this method to our example. In order to generate multiple estimates of θ_0 (which are then used to approximate both the true θ_0 and $\hat{\theta}_N$), we have used the information provided by the initial

identification which had delivered $\theta_{o,est}$. This was an open-loop identification with $\Phi_u(\omega) = 1\ \forall\omega$ and $N = 500$.[11] Using the covariance matrix of $\theta_{o,est}$, we have randomly generated 46 parameter vectors θ_i $(i = 1...46)$ around $\theta_{o,est}$; 46 samples correspond to a confidence of 80%. For each of these estimates, we have applied Theorem 7.1 and we have thus obtained 46 different lengths N_{θ_i}. A *more robust* choice of the length N is then (see Section 7.3):

$$max_{\theta_i} N_{\theta_i} = 5897$$

The standard deviation of these 46 N_{θ_i} was 383.

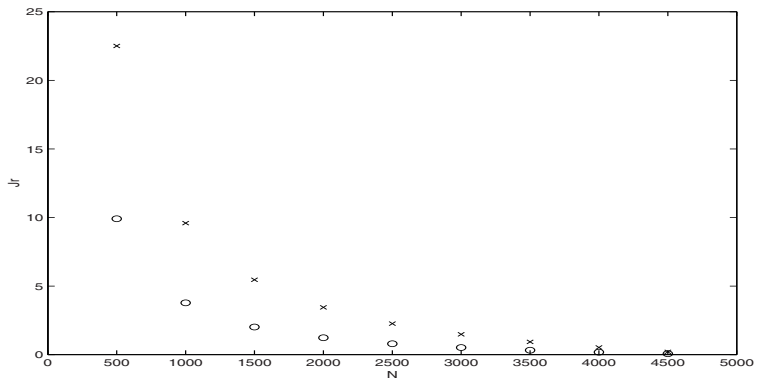

Fig. 3. Cost of the least costly experiment that satisfies Constraint 4.1 (circles) and of the white noise experiment that satisfies the same constraint (crosses), for different values of the data length.

Least costly identification with external excitation. For the same example, we have also studied the effect of applying least costly excitation signals r for a range of data lengths that were too short to lead to identifiability using only the noise excitation. Thus, we have computed the optimal signal spectra $\Phi_r(\omega)$ resulting from Theorem 7.2 for data lenghts N ranging from 500 to 4,500 by steps of 500. For each of these data lenghts, we have then compared the identification cost $\mathcal{J}_r$ resulting from the application of the optimal excitation signal (when $m = 10$ in (27)) with the cost that would result by applying a white noise reference excitation (i.e. $m = 0$ in (27)) with a variance that is sufficient to satisfy the robust performance Constraint 4.1. The comparison between the cost of the least costly experiment and the cost of a corresponding identification experiment with white noise excitation is shown in Figure 3. As can be seen from this figure, the use of an optimally designed excitation signal r reduces the identification cost by a factor of 2 to 3 whatever the data length. Similar comparisons leading to similar conclusions can be found for the case of open-loop identification in [6].

[11] This initial identification was too cheap to verify Constraint 4.1.

9 Conclusions

We have presented a new paradigm for optimal experiment design in an identification for robust control context, where the objective is to design an identification experiment at the smallest possible cost. The identification cost must be as small as possible while still delivering a model uncertainty set that just meets the robust control performance constraints. The cost of the identification experiment is expressed either as the experiment time, or in terms of a measure of the deterioration of the closed-loop performance, during the identification experiment, with respect to the closed-loop performance under normal (non perturbed) operation. This paradigm was initially proposed in [18], and subsequently elaborated upon and extended in a number of other papers [19, 21, 28]. The underlying theme of this new paradigm is "Why do more (identification) work than is needed?"

One might wonder whether it pays to apply the heavy mathematical machinery required to compute optimal input designs, given that the optimal solution necessarily depends on the unknown system, which means that a preliminary model estimate must be obtained first before an approximately optimal input signal can be computed. This is sometimes referred to as *adaptive (or iterative) optimal input design*. In [33] the benefits of optimal input design for control have been demonstrated for two benchmark problems. In both cases, significant savings are obtained by the application of a two-step identification procedure, where the second step uses an optimally designed input signal computed from a preliminary model estimate.

From a practical point of view, the cost of identification is an issue of major importance. It is often estimated that 75% of the cost associated to an advanced control project goes into model development. Even though the definition of the cost used in the present work on "least costly identification for control" does by no means cover all the practical costs of modelling, the disruption caused to normal operation and the time required to arrive at a satisfactory model are considered to be very significant elements of this total modelling cost. These two costs are incorporated in the "least costly" criterion developed in the present paper.

Acknowledgements
This paper presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors.

References

1. R. Mehra, "Optimal input signals for parameter estimation in dynamic systems - survey and new results," *IEEE Trans. on Automatic Control*, vol. AC-19, no. 6, pp. 753–768, December 1974.

2. M. Zarrop, *Optimal experiment design for dynamic system identification.* Berlin, New York: Lecture Notes in Control and Information Sciences, Vol. 21, Springer Verlag, 1979.
3. G. Goodwin and R. Payne, *Dynamic System Identification: Experiment Design and Data Analysis.* New York: Academic Press, 1977.
4. S. Karlin and W. Studden, *Tchebycheff systems with applications in analysis and statistics.* New York: Pure Appl. Math. 15, Wiley-Interscience, 1966.
5. R. Hildebrand and M. Gevers, "Identification for control: optimal input design with respect to a worst-case ν-gap cost function," *SIAM Journal on Control and Optimization*, vol. 41, no. 5, pp. 1586–1608, March 2003.
6. H. Jansson, "Experiment design with applications in identification for control," Ph.D. dissertation, Royal Institute of Technology (KTH), December 2004, tRITA-S3-REG-0404.
7. L. Ljung, "Asymptotic variance expressions for identified black-box transfer function models," *IEEE Trans. Automatic Control*, vol. AC-30, pp. 834–844, 1985.
8. B. Wahlberg and L. Ljung, "Design variables for bias distribution in transfer function estimation," *IEEE Trans. Automatic Control*, vol. AC-31, pp. 134–144, 1986.
9. M. Gevers and L. Ljung, "Optimal experiment designs with respect to the intended model application," *Automatica*, vol. 22, pp. 543–554, 1986.
10. H. Hjalmarsson, M. Gevers, and F. D. Bruyne, "For model-based control design, closed-loop identification gives better performance," *Automatica*, vol. 32, pp. 1659–1673, 1996.
11. U. Forssell and L. Ljung, "Some results on optimal experiment design," *Automatica*, vol. 36, pp. 749–756, 2000.
12. H. Jansson and H. Hjalmarsson, "A framework for mixed H_∞ and H_2 input design," in *Proc. MTNS*, Leuven, Belgium, 2004.
13. Y. Nesterov and A. Nemirovskii, *Interior Point polynomial algorithms in convex programming.* Philadelphia: SIAM Studies in Applied Mathematics, 1994, vol. 13.
14. S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia: SIAM Studies in Applied Mathematics, 1994.
15. X. Bombois, M. Gevers, G. Scorletti, and B. Anderson, "Robustness analysis tools for an uncertainty set obtained by prediction error identification," *Automatica*, vol. 37, no. 10, pp. 1629–1636, October 2001.
16. M. Gevers, X. Bombois, B. Codrons, G. Scorletti, and B. Anderson, "Model validation for control and controller validation in a prediction error identification framework - Part I: theory," *Automatica*, vol. 39, no. 3, pp. 403–415, March 2003.
17. G. Vinnicombe, "Frequency domain uncertainty and the graph topology," *IEEE Trans Automatic Control*, vol. AC-38, pp. 1371–1383, 1993.
18. X. Bombois, G. Scorletti, P. Van den Hof, and M. Gevers, "Least costly identification experiment for control: a solution based on a high-order model approximation," in *CD-ROM Proc. American Control Conference*, Boston, MA, USA, July 2004, pp. 2818–2823.
19. X. Bombois, G. Scorletti, M. Gevers, R. Hildebrand, and P. Van den Hof, "Cheapest open-loop identification for control," in *CD-ROM Proc. 33rd IEEE Conf on Decision and Control*, The Bahamas, December 2004, pp. 382–387.

20. H. Jansson and H. Hjalmarsson, "A general framework for mixed H_∞ and H_2 input design," *submitted for publication to IEEE Trans. Auto. Control*, 2004.
21. X. Bombois, G. Scorletti, Gevers, P. Van den Hof, and R. Hildebrand, "Least costly identification experiment for control," *submittted to Automatica*, 2005.
22. L. Ljung, *System Identification: Theory for the User, 2nd Edition*. Englewood Cliffs, NJ: Prentice-Hall, 1999.
23. T. Söderström and P. Stoica, *System Identification*. Hemel Hempstead, Hertfordshire: Prentice-Hall International, 1989.
24. X. Bombois, "Connecting prediction error identification and robust control analysis: a new framework," Ph.D. thesis, CESAME, Université Catholique de Louvain, Tech. Rep., 2000.
25. V. Popov, *Hyperstability of Control Systems*. New-York: Springer-Verlag, 1973.
26. K. Lindqvist, *On experiment design in identification of smooth linear systems*. Stockholm, Sweden: PhD Thesis, Royal Institute of Technology, 2001.
27. E. V. den Eijnde and J. Schoukens, "On the design of optimal excitation signals," in *Proc. 9th IFAC/IFORS Symp. on Identification and System Parameter Estimation*, Budapest, Hungary, 1991, pp. 827–832.
28. H. Jansson and H. Hjalmarsson, "Mixed H_∞ and H_2 input design for identification," in *CD-ROM Proc. 43rd IEEE Conf on Decision and Control*, The Bahamas, 2004, pp. 388–393.
29. R. Tempo, G. Calafiore, and F. Dabbene, *Randomized algorithms for analysis and control of uncertain systems*. New York: Springer Verlag, 2004.
30. R. Tempo, E. Bai, and F. Dabbene, "Probabilistic robustness analysis: explicit bound for the minimum number of samples," *Systems and Control Letters*, vol. 30, pp. 237–242, 1997.
31. I. Landau, D. Rey, A. Karimi, A. Voda, and A. Franco, "A flexible transmission system as a benchmark for robust digital control," *European Journal of Control*, vol. 1, no. 2, pp. 77–96, 1995.
32. G. Ferreres and V. Fromion, "H_∞ control for a flexible transmission system," in *CD-ROM Proc. European Control Conference*, Brussels, Belgium, 1997.
33. M. Barenthin, H. Jansson, and H. Hjalmarsson, "Applications of mixed H_2 and H_∞ input design in identification," in *16th IFAC World Congress on Automatic Control, paper 03882*, Prague, July 2005.