13th Triennial World Congress, San Francisco, USA

THE HAMBO TRANSFORM: A SIGNAL AND SYSTEM TRANSFORM INDUCED BY GENERALIZED ORTHONORMAL BASIS FUNCTIONS

Peter S.C. Heuberger [1] and **Paul M.J. Van den Hof** [2]

Mechanical Engineering Systems and Control Group, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands

Abstract. A signal and system transformation is analyzed that is induced by a recently introduced generalized orthonormal basis for $\mathcal{H}_2$-systems and ℓ_2-signals. This basis is very flexible and generalizes the pulse, Laguerre and Kautz bases. The corresponding system and signal transformations generalize the Fourier and z-transforms; interesting properties of the representations in the transform domain are shown. The transformations are indispensable in the asymptotic analysis of related system identification algorithms, and provide powerful results in system approximation.

Keywords. Orthogonal basis functions; Laguerre functions; discrete-time systems; Fourier transform; system approximation; network synthesis.

1. INTRODUCTION

The idea of decomposing representations of linear time-invariant dynamical systems and related input/output signals, e.g. with respect to their power density spectra, in terms of orthogonal components other than the standard Fourier series, dates back to the work of Lee and Wiener in the thirties, as reviewed in Lee (1960). Laguerre functions have been very popular in this respect, mainly because of the fact that their frequency response is rational. In an attempt to find more general classes of orthogonal basis functions with this same property, Kautz (1954) formulated a general class of functions, composed of damped exponentials, to be used for signal decomposition.

In the seventies and eighties, particularly Laguerre functions were often applied in problems of network synthesis, system approximation and identification (King and Paraskevopoulos, 1979; Nurges 1987). In some cases a system transformation in terms of the Laguerre basis functions has been considered here. Later, in Wahlberg (1991, 1994a,1994b) Laguerre functions and so-called two-parameter Kautz functions have been used in the identification of the expansion coefficients of approximate models by simple linear regression methods, while system approximation properties are discussed in Wahlberg and Mäkilä (1996).

Generalizing the basis functions, Heuberger (1991) has developed a theory on the construction of orthogonal basis functions, based on balanced realizations of inner (all-pass) transfer functions, see Heuberger *et al.* (1995). The construction of these functions generalizes the Laguerre and two-parameter Kautz case. This development has led to a generalization of the identification results of Wahlberg, see Van den Hof *et al.* (1995). A closely related approach to incorporate general Kautz functions

[1] The work of Peter Heuberger is supported by the Dutch Technology Foundation (STW) under contract DWT55.3618.
[2] Author to whom correspondence should be addressed; E-mail: P.M.J.vandenHof@wbmt.tudelft.nl. The research reported here has been supported by the Commission of the European Community, Human Capital and Mobility Program, network SIMONET.

into the identification framework is discussed in Ninness and Gustafsson (1994) and Ninness *et al.* (1995).

Besides the use of these functions for identification purposes, the basis functions of Heuberger *et al.* (1995) give rise to a general theory on dynamical signals and systems transformations induced by these so-called Hambo basis functions. Properties of these transformations have been crucial in the development of the identification results in Van den Hof *et al.* (1995).
In this paper, the mentioned transform-theory will be presented, and its relevance will be shown to a related system approximation problem. For the proofs of all results the reader is referred to Heuberger and Van den Hof (1995). Interesting results on system approximation using the same basis functions are discussed in Oliveira e Silva (1996).

The core of the paper is the layout of Figure 1 showing the so called *Hambo transformations* for ℓ_2 signals in the time and the frequency domain. In this figure ϕ and ψ reflect the signal transform and its inverse; the related transform of dynamical systems is indicated by the transform H and its inverse H^{-1}. In this system transform the ℓ_2-signal x is treated as the pulse response of the dynamical system.

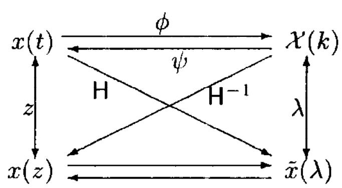

Fig. 1. Commuting diagram showing the Hambo-transform and the Inverse Hambo-transform, defined on ℓ_2-signals.

2. THE HAMBO BASIS FUNCTION

For the construction of the basis functions the following nesting property of balanced realizations of inner functions is employed.

Proposition 2.1 *(Roberts and Mullis, 1987; Heuberger et al., 1995.) Let G_b be a square inner transfer function with minimal balanced realization (A_1, B_1, C_1, D_1) having state dimension $n_b > 0$. Then for any $k > 1$ the realization (A_k, B_k, C_k, D_k) with*

$$A_k = \begin{bmatrix} A_{k-1} & 0 \\ B_1 C_{k-1} & A_1 \end{bmatrix} \quad B_k = \begin{bmatrix} B_{k-1} \\ B_1 D_1^{k-1} \end{bmatrix} \tag{1}$$

$$C_k = \begin{bmatrix} D_1^{k-1} C_1 & C_{k-1} \end{bmatrix} \quad D_k = D_1 \cdot D_{k-1}. \tag{2}$$

is a minimal balanced realization of G_b^k with state dimension $n_b \cdot k$.

Given a balanced realization of G_b one can directly construct a balanced realization of G_b^k for any $k > 0$ with this recursive mechanism. By writing down the equation for the state trajectory related to the realization (A_k, B_k, C_k, D_k):

$$x_k(t+1) = A_k x_k(t) + B_k u(t); \tag{3}$$

it follows from the structure of A_k and B_k that for any input and initial state, $x_k(t)$ can be written as

$$x_k(t) = \begin{bmatrix} x_{k-1}(t) \\ \phi_k(t) \end{bmatrix}, \tag{4}$$

and thus

$$x_k(t) = \begin{bmatrix} \phi_1^T(t) & \phi_2^T(t) & \cdots & \phi_k^T(t) \end{bmatrix}^T. \tag{5}$$

The main result of concern is that, if in (3) $u(t) = \delta(t)$ and $x_k(0) = 0$, the sequence of $\ell_2[1, \infty)$-functions

$$\{e_i^T \cdot \phi_k(t)\}_{i=1\cdots,n_b;k=1,\cdots,\infty} \tag{6}$$

is an orthonormal basis for the Hilbert space $\ell_2[1, \infty)$.

As in the case of orthonormal basis functions based on e.g. Laguerre and Kautz functions, the introduced basis incorporates dynamics, present in the inner function G_b. As the McMillan degree of G_b is not limited, the complexity of the dynamics, present in the basis, can be arbitrary.

It is a particular choice to structure this basis in terms of the n_b-dimensional components $\phi_k(t)$. This is motivated by the following shift structure:

$$\phi_{k+1}(t) = G_b(q) I_{n_b} \cdot \phi_k(t) \quad k = 1, 2, \cdots \tag{7}$$

$$\phi_1(t) = A^{t-1} B \tag{8}$$

where the shift operator q operates on the time sequence ϕ_k, and $\phi_k(t) = 0$ for $t \leq 0$. By z-transform it follows that the sequence of $\mathcal{H}_2$-functions determined by the entries of

$$V_k(z) := \sum_{t=1}^{\infty} \phi_k(t) z^{-t} = (zI - A)^{-1} B G_b^{k-1}(z) \tag{9}$$

constitute an orthonormal basis for the Hilbert space of strictly proper stable systems in $\mathcal{H}_2$.

As a result, for any strictly proper system $H(z) \in \mathcal{H}_2$ or signal $y(t) \in \ell_2[1, \infty)$ there exist unique series expansions:

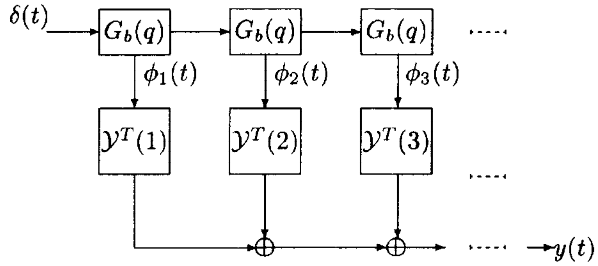

Fig. 2. Representation of ℓ_2-signal in terms of the ℓ_2 basis functions $\phi_k(t)$.

$$H(z) = \sum_{k=1}^{\infty} L_k^T V_k(z) \quad L_k \in \mathbb{R}^{n_b \times 1}, \tag{10}$$

$$y(t) = \sum_{k=1}^{\infty} \mathcal{Y}^T(k)\phi_k(t) \quad \mathcal{Y}(k) \in \mathbb{R}^{n_b \times 1}. \tag{11}$$

The above construction is depicted in the network that is sketched in Figure 2. In the network, the arrows departing from midway block locations refer to balanced state readout.

For specific choices of $G_b(z)$ well known classical basis functions can be generated. $G_b(z) = z^{-1}$ results in the standard pulse basis $V_k(z) = z^{-k}$ and the first order inner function $G_b(z) = \frac{1-az}{z-a}$ results in the Laguerre basis $V_k(z) = \sqrt{1-a^2}\frac{(1-az)^k}{(z-a)^{k+1}}$. Similarly the two-parameter Kautz functions originate from the choice of a second order inner function.

The introduced basis induces a signal transform $y \to \eta$ with $[\mathcal{Y}^T(1)\ \mathcal{Y}^T(2)\ \cdots] = [\eta(1)\ \eta(2)\ \cdots]$ where $\eta(t)$ is scalar valued. Before analyzing this basis and the corresponding transform further, we will first discuss a dual basis in ℓ_2 that induces the corresponding inverse transform $\eta \to y$.

3. THE DUAL BASIS IN ℓ_2

A basis for ℓ_2 that is dual to the basis presented in the previous section, is given in the next proposition.

Proposition 3.1 *Denote*

$$\psi_k(t) := \phi_t(k) \quad \textit{for } t, k = 1, ..., \infty \tag{12}$$

$$\gamma_k((t-1)n_b + i) := e_i^T \psi_k(t), \quad i = 1, \cdots, n_b. \tag{13}$$

i.e. $[\psi_k^T(1)\ \psi_k^T(2)\ \cdots] = [\gamma_k(1)\ \gamma_k(2)\ \gamma_k(3)\ \cdots]$.

Then the ℓ_2-signals γ_k constitute an orthonormal basis for $\ell_2[1,\infty)$, which is dual to the basis in the previous section, in the sense that for each $y \in \ell_2[1,\infty)$ there is a transform $\eta \in \ell_2[1,\infty)$ given by

$$\eta(t) = \sum_{k=1}^{\infty} y(k)\gamma_k(t). \tag{14}$$

For the original basis reflected by $\phi_k(t)$ we formulated in (7,8) a nice shift structure of the basis functions. A similar result for the dual basis is formulated next.

Proposition 3.2 *Let the inner function G_b have a minimal balanced realization (A, B, C, D). Then (D, C, B, A) is a minimal balanced realization of the $n_b \times n_b$ inner function $N(z)$ with McMillan degree equal to 1, where*

$$N(z) := A + B(z-D)^{-1}C \tag{15}$$

Furthermore,

$$\psi_{k+1}(t) = N(q) \cdot \psi_k(t) \quad k = 1, 2, \cdots \tag{16}$$

$$\psi_1(t) = BD^{t-1} \tag{17}$$

where the shift operator q operates on the time sequence ψ_k, and $\psi_k(t) = 0$ for $t \leq 0$.

The Proposition shows that the inner functions G_b and N clearly play a dual role. They are simply related by ordering the state space realizations reversely.
A similar duality can be considered between the signal sequences $\phi_k(t)$ and $\psi_k(t)$. Whereas $\phi_k(t)$ originates from the balanced states of G_b^k under pulse excitation and zero initial conditions, $\psi_k(t)$ is the output of N^k under zero excitation and initial condition $x_k(0) = e_1$, where $x_k(t)$ is the balanced state.

The construction of the $\ell_2^{n_b}[1,\infty)$-signals $\psi_k(t)$ is depicted in the network of Figure 3. This network also shows how the transform $y \to \eta$ as discussed in the previous section can be calculated. In this respect the network is dual to the network depicted in Figure 2.

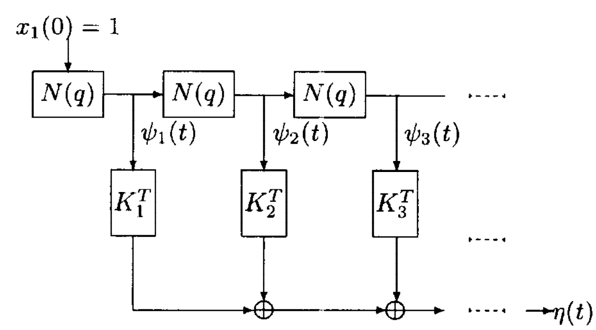

Fig. 3. Network showing the construction of $\psi_k(t)$ related to the dual basis, for calculation of the ℓ_2-signal transform $y \to \eta$, $\eta(t) = \sum_{k=1}^{\infty} K_k^T \psi_k(t)$, where $K_k^T = [y((k-1)n_b + 1) \cdots y(kn_b)]$.

Similar to (9) we will denote the z-transform of the functions $\psi_k(t)$ by

$$W_k(z) := \sum_{t=1}^{\infty} \psi_k(t) z^{-t} \tag{18}$$

while as a direct result of Proposition 3.2 it holds that

$$W_k(z) = N^{k-1}(z) \cdot (z - D)^{-1} B. \tag{19}$$

It has to be noted that, whereas the scalar functions $\{e_i^T V_k(z)\}$ constitute an orthonormal basis, the dual form of this basis is *not* given by $\{e_i^T W_k(z)\}$. This is due to the specific relation (13) which shows that a reshuffling of the components of $\psi_k(t)$ over time is required to arrive at the orthonormal (dual) basis.

4. THE HAMBO SIGNAL TRANSFORM

The ℓ_2-basis functions presented in the previous sections generate a signal transformation $\ell_2[1,\infty) \to \ell_2[1,\infty)$. However, it is particularly advantageous to consider a closely related signal transformation that directly uses the n_b-dimensional signals $\phi_k(t)$ and $\psi_k(t)$. This is mainly due to the nice shift structures that these functions satisfy, as formulated in the equations (7,8) and Proposition 3.2. This shift structure enables the construction of a dynamical system transformation in terms of rational functions, as will be discussed in section 5.

Definition 4.1 *Let $\{\phi_k(t)\}_{k=1,\cdots\infty}$ be a sequence of $\ell_2^{n_b}[1,\infty)$-functions, being generated by an inner function G_b with McMillan degree n_b as presented in Section 2. Then we define the Hambo-transform as the mapping H: $\ell_2^m[1,\infty) \to \mathcal{H}_2^{n_b \times m}$, determined by*

$$\mathsf{H}(x) := \tilde{x}(\lambda) = \sum_{k=1}^{\infty} \mathcal{X}(k) \lambda^{-k} \tag{20}$$

with the Hambo coefficients $\mathcal{X}(k)$, determined by

$$\mathcal{X}(k) := \sum_{t=1}^{\infty} \phi_k(t) x^T(t). \tag{21}$$

Through this transformation, vector ℓ_2-signals are transformed to matrix-valued sequences. This Hambo transform can be considered as a generalization of the Fourier or the z-transform, the latter of which for a signal $x \in \ell_2[1,\infty)$ is given by $x(z) = \sum_{t=1}^{\infty} x(t) z^{-t}$. This z-transform is generated by (20) employing the orthonormal (pulse) basis, $\phi_k(t) = \delta(k-t)$, corresponding to $G_b(z) = z^{-1}$.

Some basic properties of this Hambo transform are collected in the following Proposition.

Proposition 4.2 *The Hambo transform as defined in Definition 4.1 satisfies*

(a) $\tilde{x}(\lambda) = \sum_{k=1}^{\infty} W_k(\lambda) x^T(k)$.

(b) $\tilde{x}(\lambda) = [x^T(z)z]_{z^{-1}=N(\lambda)} \cdot [I_m \otimes W_1(\lambda)]$ *which for scalar x reduces to* $\tilde{x}(\lambda) = [x(z)z]_{z^{-1}=N(\lambda)} \cdot W_1(\lambda)$.

Apparently this signal transform can be obtained by a simple variable-transformation $z^{-1} \to N(\lambda)$. The Hambo transform also yields an inverse transform, formulated next.

Proposition 4.3 *The inverse Hambo transform H^{-1} : $\mathcal{H}_2^{n_b \times m} \to \ell_2^m$ is defined by*

$$\mathsf{H}^{-1}(\tilde{x})(t) := \sum_{k=1}^{\infty} \mathcal{X}^T(k) \psi_t(k) = x(t) \tag{22}$$

$$\textit{with} \quad \mathcal{X}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{x}(e^{i\omega}) e^{ik\omega} d\omega. \tag{23}$$

Dual to Proposition 4.2 the following results can be formulated for the inverse transform.

Proposition 4.4 *The Inverse Hambo transform as defined above satisfies*

$$x^T(z) = \sum_{k=1}^{\infty} V_k^T(z) \mathcal{X}(k) = V_1^T(z) \cdot [\tilde{x}(\lambda)\lambda]_{\lambda^{-1}=G_b(z)}$$

The diagram shown in Figure 1 sketches the different transformations that have been considered. The relations indicated by ϕ and ψ are defined in (21) and (22). The relations indicated by H and H^{-1} are defined in (20) and (22), whereas the direct relations between $x(z)$ and $\tilde{x}(\lambda)$ are provided in Propositions 4.2 and 4.4.

5. THE HAMBO SYSTEM TRANSFORM

The Hambo transform of ℓ_2-signals, as introduced in the previous section, induces also a linear system transformation. This transformed system describes the relationship between (transformed) input and output signals.

Proposition 5.1 *Let $P \in \mathcal{H}_2$ and let $u, y \in \ell_2$ such that $y(t) = P(q)u(t)$. Consider the Hambo transform of ℓ_2 signals as defined in definition 4.1. Then there exists a $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$ satisfying*

$$\tilde{y}(\lambda) = \tilde{P}(\lambda) \tilde{u}(\lambda). \tag{24}$$

The mapping T: $\mathcal{H}_2 \to \mathcal{H}_2^{n_b \times n_b}$ defined by $\mathsf{T}(P) := \tilde{P}(\lambda)$ is referred to as the Hambo system-transform, *and the* *inverse mapping* T^{-1} *is denoted as the* inverse Hambo system-transform.

Proposition 5.2 *Let P be written as* $P(z) = \sum_{k=0}^{\infty} p_k z^{-k}$. *Then the Hambo system-transform* $\mathsf{T}(P)$ *is determined by*

$$\tilde{P}(\lambda) = P(z)|_{z^{-1}=N(\lambda)} = \sum_{k=0}^{\infty} p_k N(\lambda)^k, \qquad (25)$$

The Hambo-transform of any system P can be obtained by a simple variable-transformation $z^{-1} = N(\lambda)$.

In terms of the sequence of expansion coefficients, equation (24) shows that

$$\mathcal{Y}(k) = \tilde{P}(q)\mathcal{U}(k) \quad \text{for all } k \geq 1 \qquad (26)$$

where $\mathcal{Y}, \mathcal{U}$ are $\ell_2^{n_b}[1,\infty)$ sequences induced by the Hambo transforms of y, u, and the shift operator q operates on the sequence index k. This result generalizes the situation of a corresponding Laguerre transformation, where it concerns the variable-transformation $z = \frac{\lambda+a}{1+a\lambda}$ (see also Wahlberg, 1991).

The basis generating inner function G_b itself transforms to a simple shift in the Hambo-domain:

$$\tilde{G}_b(\lambda) = \lambda^{-1} I_{n_b}$$

The complex (matrix) function $\tilde{P}(\lambda)$ provides an alternative representation of the dynamical system P. Many of the system theoretic properties of $P(z)$ carry over to $\tilde{P}(\lambda)$. However there are also important differences that will be clarified in the sequel.

Most properties of the Hambo system-transform can be derived from state-space realizations of the transformed system $\tilde{P}$. See Heuberger and Van den Hof (1995) for explicit formula's of this realization.

One of the main properties is that $\tilde{P}$ and P have the same McMillan degree. The poles and zeros of P and $\tilde{P}$ also have close relationships.

Proposition 5.3 *Let* $\tilde{P}$ *be the Hambo system-transform of a scalar dynamical system* $P \in \mathbb{R}\mathcal{H}_2$, *induced by the inner function* G_b *and let* $G_b(z)$ *have poles* ρ_j, $j = 1, \cdots n_b$. *If* $P(z)$ *has poles (zeros) in* $z = \alpha_i$, *then* $\tilde{P}(\lambda)$ *has poles (zeros) in*

$$\lambda_i = G_b(\alpha_i^{-1}) = \prod_{j=1}^{n_b} \frac{\alpha_i - \rho_j}{1 - \alpha_i \rho_j} \qquad (27)$$

Finally, we will specify how the inverse Hambo system transform can be calculated.

Proposition 5.4 *Let* $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$. *Then the inverse Hambo system-transform* $\mathsf{T}^{-1}(\tilde{P})$ *is determined by*

$$P(z) = zV_1^T(z) \cdot \tilde{P}(\lambda)|_{\lambda^{-1}=G_b(z)} \cdot \frac{B}{1 - DG_b(z)} \qquad (28)$$

In correspondence with the forward transform, the inverse transform maps a system in $\mathcal{H}_2^{n_b \times n_b}$ back to a scalar system.

The Hambo system transform exhibits several more nice properties, as e.g. invariance properties of Hankel singular values and several norms as the $\mathcal{H}_\infty$-norm and the $\mathcal{H}_2$-norm (Heuberger, 1991).

6. HAMBO TRANSFORM IN SYSTEM APPROXIMATION

The most straightforward use of the signal and system transformations discussed in this paper, is in the area of system approximation. Suppose we have been given a scalar stable and strictly proper dynamical system $P(z)$, then we can represent this system in the series expansion:

$$P(z) = \sum_{k=1}^{\infty} L_k^T V_k(z). \qquad (29)$$

For an unknown system $P(z)$, we can also identify the system based on a parametrization in terms of the sequence $\{L_k\}_{k=1,\cdots}$ as is analyzed in Van den Hof *et al.* (1995).

For an appropriate choice of the inner function G_b, the basis functions $V_k(z)$ should match the most dominant components of $P(z)$ such that the series expansion will have a high rate of convergence. In other words: for a given approximate model

$$\hat{P}_n(z) := \sum_{k=1}^{n} L_k^T V_k(z) \qquad (30)$$

the approximation error $\|P - \hat{P}_n\|$ (in some norm), will be dependent on the choice of $V_k(z)$.

We will now show that we can explicitly relate the rate of convergence of this series expansion to the dynamics that is present in P and G_b, by using the transform results discussed previously.

If $p(t)$ is the pulse response related to $P(z)$, then $p(t) = \sum_{k=1}^{\infty} L_k^T \phi_k(t)$, which implies with the signal transform definitions that

$$\tilde{p}(\lambda) = \sum_{k=1}^{\infty} L_k \lambda^{-k}.$$

In other words, the decay rate of the sequence $\{L_k\}_{k=1,\cdots}$ is governed by the dynamics that is present in $\tilde{p}(\lambda)$.

Proposition 6.1 *$\tilde{p}(\lambda) = \tilde{P}(\lambda)\frac{C^T}{1-\lambda D}$ and the poles of $\tilde{p}$ are given by the poles of $\tilde{P}$.*

This result leads to the following Proposition.

Proposition 6.2 *Let P have poles α_i, $i = 1, \cdots, n_g$, and let $G_b(z)$ have poles ρ_j, $j = 1, \cdots n_b$. Denote*

$$\mu := \max_i \prod_{j=1}^{n_b} \left| \frac{\alpha_i - \rho_j}{1 - \alpha_i \rho_j} \right| = \max_i |G_b(\alpha_i^{-1})| \quad (31)$$

Then there exists a constant $c \in I\!R$ such that for all $\zeta > \mu$

$$\|P - \hat{P}_n\|_2 \leq c \cdot \frac{\zeta^{n+1}}{\sqrt{1-\zeta^2}}. \quad (32)$$

If P(z) has only poles with multiplicity 1, a more explicit bound can be calculated as is also shown in Ninness *et al.* (1995). In this case $P(z)$ can be written in the fractional expansion

$$P(z) = \sum_{i=1}^{r} \frac{k_i}{z - \alpha_i}$$

and it can be shown that such a system obeys

$$P(z) - \hat{P}_n(z) = \sum_{i=1}^{r} \frac{k_i G_b^n(\alpha_i^{-1})}{z - \alpha_i} G_b^n(z) \quad (33)$$

$$\|P - \hat{P}_n\|_2 \leq \mu^n \sum_{i=1}^{r} \frac{|k_i|}{\sqrt{1-|\alpha_i|^2}} \quad (34)$$

These results show that an appropriate choice of basis, can drastically improve the rate of convergence in the series expansion, and thus enabling more accurate system approximations with fewer terms.

7. CONCLUSIONS

We have analyzed a signals and systems transform that is induced by a very general class of orthogonal functions. The basis functions are induced by the balanced states of scalar inner (stable all-pass) functions, and generalize the classical Laguerre and Kautz functions. The induced signals and systems transforms generalize the Fourier and z-transform to a multidimensional representation. The benefit of the transformations in a related system approximation problem has been shown.

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