Basis Functions Induced by Balanced Realizations of Inner Functions, and their Role in System Approximations ‡

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The idea of decomposing representations of linear time-invariant dynamical systems and their related signals in terms of orthogonal components other than the standard Fourier series, dates back to the work of Lee and Wiener in the thirties, as reviewed in Lee (1960). Laguerre functions have been very popular in this respect, mainly because of the fact that their frequency response is rational (Gottlieb, 1938; Schetzen, 1970).
In an attempt to find more general classes of orthogonal basis functions with this same property, Kautz (1954) formulated a general class of functions, composed of damped exponentials, to be used for signal decomposition.
In the seventies and eighties, particularly Laguerre functions were often applied in problems of network synthesis, system approximation and identification. In some cases a system transformation in terms of the Laguerre basis functions has been considered here. Later, in Wahlberg (1991, 1994a,1994b) Laguerre functions and so-called two-parameter Kautz functions have been used in the identification of the expansion coefficients of approximate models by simple linear regression methods.
Extending this work further, Heuberger (1991) has developed a theory on the construction of orthogonal basis functions, based on balanced realizations of inner (all-pass) transfer functions, see Heuberger *et al.*(1995). The construction of these functions generalizes the

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Laguerre and two-parameter Kautz case. This development has led to a generalization of the identification results of Wahlberg, see Van den Hof *et al.*(1994).
Further attempts to incorporate general Kautz functions into the identification framework are discussed in Ninness and Gustafsson (1994).
Besides the use of these functions for identification purposes, the basis functions of Heuberger *et al.* (1995) give rise to a general theory on dynamical signals and systems transformations induced by these so-called Hambo basis functions. Properties of these transformations have been crucial in the development of the identification results in Van den Hof *et al.* (1994).
In this paper we will discuss the relevance of the mentioned basis functions for problems of system approximation. It will appear that a signal and systems transformation induced by these basis functions is instrumental in deriving approximation results.
We will first sketch the main properties of the basis functions considered.

Theorem 0.1 (Heuberger *et al.*, 1995) *Let $G_b(z)$ be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A, B, C, D). Let the balanced states of the systems G_b^k be denoted by $x_k(t)$ for $t, k \in \mathbf{Z}_+$. Then*

(a) Under pulse input conditions, i.e. $u(t) = \delta(t+1)$, the sequence of ℓ_2-functions $\{e_i^T \cdot x_k(t)\}_{i=1,\cdots,n_b k}$ constitutes an orthonormal basis for the Hilbert space $\ell_2[0,\infty)$ when $k \to \infty$;

(b) The sequence of rational functions in $\mathbb{R}\mathcal{H}_2$ given by $\{e_i^T \cdot V_k(z)\}_{i=1,\cdots,n_b}$ with $V_k(z) = z(zI-A)^{-1}B \cdot G_b(z)^k$, constitutes an orthonormal basis for the Hilbert space of stable systems $\mathcal{H}_2$. □

With the notation $V_k(z) = \sum_{t=0}^{\infty} \phi_k(t) z^{-t}$ it follows that the related functions $\phi_i \in \ell_2^{n_b}[0,\infty)$ satisfy

$$\phi_k(t) = G_b(q) I_{n_b} \cdot \phi_{k-1}(t) \quad k = 1, 2, \cdots \tag{1}$$

$$\phi_0(t) = A^t B \tag{2}$$

$$\phi_t(k) = N(q) \cdot \phi_t(k-1) \tag{3}$$

$$\phi_t(0) = BD^t \tag{4}$$

where the shift operator q operates on the index t, $N(z)$ $n_b \times n_b$ inner function with minimal balanced realization (D, C, B, A), and $\phi_k(t) = 0$ for $t < 0$.
While the scalar components of ϕ_i consitute an orthonormal basis of ℓ_2, for each $y \in \ell_2$ there exists a unique expansion: $y(t) = \sum_{k=0}^{\infty} \mathcal{Y}^T(k) \phi_k(t)$, $\mathcal{Y}(k) \in \mathbb{R}^{n_b \times 1}$. This expansion gives rise to a signal transformation, as defined next.

Definition 0.2 *We define the Hambo-transform as the mapping H: $\ell_2^m \to \mathcal{H}_2^{n_b \times m}$, determined by*

$$\mathsf{H}(x) := \tilde{x}(\lambda) = \sum_{k=0}^{\infty} \mathcal{X}(k) \lambda^{-k} \tag{5}$$

with the Hambo coefficients $\mathcal{X}(k) \in \ell_2^{n_b \times m}[0, \infty)$*, determined by* $\mathcal{X}(k) := \sum_{t=0}^{\infty} \phi_k(t) x^T(t)$*.*
□

Through this transformation, ℓ_2-signals are transformed to a transform domain. This Hambo transform can be considered as a generalization of the Fourier or the z-transform, the latter of which for a signal $x \in \ell_2$ is given by $x(z) = \sum_{t=0}^{\infty} x(t) z^{-t}$. This z-transform is generated by (5) employing the orthonormal (pulse) basis, $\phi_k(t) = \delta(k - t)$, corresponding to $G_b(z) = z^{-1}$.
The Hambo transform of ℓ_2-signals, as introduced here, induces also a linear system transformation. This transformed system describes the dynamical relationship between (transformed) input and output signals.

Proposition 0.3 *Let* $P \in \mathcal{H}_2$ *and let* $u, y \in \ell_2$ *such that* $y(t) = P(q)u(t)$*. Consider the Hambo transform of* ℓ_2 *signals as defined in definition 0.2. Then there exists a* $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$ *satisfying* $\tilde{y}(\lambda) = \tilde{P}(\lambda)\tilde{u}(\lambda)$*. The mapping* $\mathsf{T}\colon \mathcal{H}_2 \to \mathcal{H}_2^{n_b \times n_b}$ *defined by* $\mathsf{T}(P) := \tilde{P}(\lambda)$ *is referred to as the* Hambo system-transform*.* □

Proposition 0.4 *Consider the situation of Proposition 0.3. Let* P *be written as* $P(z) = \sum_{k=0}^{\infty} p_k z^{-k}$*. Then the Hambo system-transform* $\mathsf{T}(P)$ *is determined by* $\tilde{P}(\lambda) = \sum_{k=0}^{\infty} p_k N(\lambda)^k$*, or differently denoted:* $\tilde{P}(\lambda) = P(z)|_{z^{-1}=N(\lambda)}$*.* □

The Hambo-transform of any system P can be obtained by a simple variable-transformation on the original transfer function, where the variable transformation concerned is given by $z^{-1} = N(\lambda)$.
Note that this result generalizes the situation of a corresponding Laguerre transformation, where it concerns the variable-transformation $z = \dfrac{\lambda + a}{1 + a\lambda}$ (see also Wahlberg, 1991). However due to the fact that in our case the McMillan degree of the inner function that generates the basis is $n_b \geq 1$, the Hambo-transformed system $\tilde{P}$ increases in input/output-dimension to $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$, as $N(\lambda)$ is an $n_b \times n_b$ rational transfer function matrix of McMillan degree 1 (since D is scalar).
It can be shown that many system properties are invariant under a Hambo system transform, as e.g. the McMillan degree, Hankel singular values and the $\mathcal{H}_\infty$-norm. The poles and zeros of P and $\tilde{P}$ also have close relationships.

Proposition 0.5 *Let* $\tilde{P}$ *be the Hambo system-transform of a scalar dynamical system* $P \in \mathbb{R}\mathcal{H}_2$*, induced by the inner function* G_b*.*

(a) If $P(z)$ *has a stable pole in* $z = \alpha$*, then* $\tilde{P}(\lambda)$ *has a stable pole in* $\lambda = G_b(\alpha^{-1})$*.*

(b) If $P(z)$ *has a zero in* $z = \beta$*, then* $\tilde{P}(\lambda)$ *has a zero in* $\lambda = G_b(\beta^{-1})$*.* □

The most straightforward use of the signal and system transformations discussed in this paper, is in the area of system approximation. Suppose we have been given a scalar stable and strictly proper dynamical system $P(z)$, then we can represent this system in the series expansion:

$$P(z) = z^{-1}\sum_{k=0}^{\infty} L_k^T V_k(z). \tag{6}$$

For an appropriate choice of the inner function G_b, the basis functions $V_k(z)$ should match the most dominant components of $P(z)$ such that the series expansion will have a high speed of convergence. In other words: for a given approximate model

$$\hat{P}_n(z) := z^{-1}\sum_{k=0}^{n-1} L_k^T V_k(z) \tag{7}$$

the approximation error $\|P(z) - \hat{P}_N(z)\|$ (in some norm), will be dependent on the choice of $V_k(z)$.

We will now show that we can explicitly relate the speed of convergence of this series expansion to the dynamics that is present in P and G_b, by using the transform results discussed previously.

If $p(t)$ is the impulse response related to $P(z)$, then $p(t) = \sum_{k=0}^{\infty} L_k^T \phi_k(t)$, which implies with the signal transform definitions that

$$\tilde{p}(\lambda) = \sum_{k=0}^{\infty} L_k \lambda^{-k}.$$

In other words, the decay rate of the sequence $\{L_k\}_{k=0,\cdots}$ is governed by the dynamics that is present in $\tilde{p}(\lambda)$.

Proposition 0.6 *Let P have poles α_i, $i = 1, \cdots, n_g$, and let $G_b(z)$ have poles ρ_j, $j = 1, \cdots n_b$. Denote $\mu := \max_i \prod_{j=1}^{n_b} \left|\frac{\alpha_i - \rho_j}{1-\alpha_i\rho_j}\right|$. Then there exists a constant $c \in I\!R$ such that for all $\zeta > \mu$*

$$\|P(z) - \hat{P}_n(z)\|_2 \leq c \cdot \frac{\zeta^{n+1}}{\sqrt{1-\zeta^2}}. \tag{8}$$

This result shows that an appropriate choice of basis, can drastically improve the speed of convergence in the series expansion, and thus enabling more accurate system approximations with fewer terms. In case G_b is strictly proper, it can be shown that the system transform has a contraction property, inducing that the poles that govern the series expansion are guaranteed to be faster than the original ones (smaller amplitude).

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