Least-Costly Experiment Design for Uni-Parametric Linear Models: An Analytical Approach

M.G. Potters, M. Forgione, X. Bombois, and P.M.J. Van den Hof

Abstract— **Least-costly experiment design has received ample attention over the past decades, and efficient numerical algorithms that can compute optimal excitation spectra for linear models have been found. The interpretation of such spectra, however, has received far less attention. We restrict ourselves to uni-parametric models, for which an analytical solution to the experiment design problem is derived. This solution enables us to address, among other things, the following questions: What determines the frequency and amplitude of the excitation signal? Does the optimal frequency depend on the location(s) that the parameter occupies in the transfer function? With the optimal signal, is a closed-loop identification experiment cheaper than an open-loop one? We show that the answers are quite delicate for some of these questions.**

I. INTRODUCTION

The design of optimal identification experiments for control has received a lot of attention in recent years [2], [4], [5], [8]. In the least-costly paradigm, the problem is formulated as an optimization problem where the decision variable is the power spectrum Φ_r of the excitation signal. The optimal spectrum is the one that minimizes the cost of the identification while guaranteeing that the uncertainty of the to-be-identified model is small enough to enable robust control design with satisfying performance. Adopting the framework of [4] reduces the problem to one in which the cost of the identification has to be minimized, while ensuring that the inverse P_θ^{-1} of the covariance matrix of the to-be-identified parameter vector is larger than a positive and symmetric matrix R_{adm}. This matrix can be computed based on user-imposed control specifications.

The cost of an identification experiment is generally expressed as a linear function of the spectrum Φ_r, and the inverse of the covariance matrix is affine in Φ_r (in the case of a linear, explicitly-known controller). Therefore, the optimal experiment design problem is convex and can thus be solved using standard convex optimization techniques.

Recent theoretical and numerical developments have resulted in the ability to solve the optimization problem for linear systems with nonlinear or implicit controllers as well, see e.g. [5] and [8]. This has broadened the range of applications considerably.

Apart from some technicalities, this least-costly optimization problem is now solvable for essentially all linear systems parameterized by an arbitrary number of parameters and with any type of controller, through the use of convex optimization algorithms.

A remaining challenge from a theoretical point of view is, however, the interpretation of the optimal power spectra that result from these numerical algorithms. The ability to derive analytically the solution to the least-costly problem will enhance our understanding, and could help in speeding up numerical algorithms. Obviously, this ability will be limited to specific cases. We restrict attention in this paper to LTI systems where only one parameter has to be identified. (The parameter may be located at multiple locations in the transfer function.) This situation occurs in e.g. reservoir engineering [7] and in functional magnetic resonance imaging [1].

For this particular class of systems, we derive an analytical solution of the least-costly identification experiment design problem (formulated in the framework of [4]). We consider both the open-loop and closed-loop identification cases and we show that the unique solution of the least-costly identification experiment problem is a single sinusoid with a specific amplitude and frequency. The fact that the optimal solution is a sinusoid could have been derived from the result in [9, pages 35-36]. However, its frequency is there only given for a very special case due to normalization.

Since we have an analytical solution for this special case, it is now possible to interpret the result. As an example, we provide insight into the frequency and amplitude of the optimal excitation signal. Furthermore, we derive that for some particular family of cost functions it is always better to do an identification experiment in closed loop compared to open loop. It is also shown that when the to-be-identified parameter resides at just one location in the transfer function, the optimal frequency only depends, location-wise, on whether the parameter is located in the numerator or denominator.

The results above pertains to the case where the cost of the identification is a linear function of the power spectrum of the excitation. We also present a method to solve the optimization problem when the cost is a nonlinear function of the spectrum, a case for which known numerical methods are no longer applicable.

The paper is organized as follows. In Section II, we introduce the notations and the least-costly optimization

M.G. Potters, M. Forgione and X. Bombois are with Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands {m.g.potters,m.forgione, x.j.a.bombois}@tudelft.nl

P.M.J. Van den Hof is with the Department of Electrical Engineering, Eindhoven University of Technology, Den Dolech 2, Eindhoven, The Netherlands p.m.j.vandenhof@tue.nl

The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no 257059, The 'Autoprofit' project (www.fp7-autoprofit.eu).

Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.

problem under consideration. Our analytical solutions are derived in Section III. We discuss and interpret our findings in Section IV, and provide a numerical study in Section V. Lastly, we draw conclusions in Section VI.

II. LEAST-COSTLY IDENTIFICATION EXPERIMENT

Consider a linear data-generating system consisting of a uni-parametric model $G_0 = G(z,\theta_0)$ with unknown parameter $\theta_0 \in \mathbb{R}$ and a known noise model $H_0 = H(z)$. The parameter θ_0 may be present at an arbitrary number of positions in the transfer function $G(z,\theta_0)$. We will consider the identification of this unique parameter in both an open-loop and in a closed-loop setting. Since the open-loop setting is a special case of the closed-loop one, we first present the closed-loop identification setting. We thus suppose that the true system is operated in closed loop with a known linear controller $C(z)$ (see Fig. 1). The output $y(t) \in \mathbb{R}$ and input $u(t) \in \mathbb{R}$ of the true system are thus given by

$$u(t) = r(t) - C(z)y(t), \quad (1)$$
$$y(t) = G_0(z)u(t) + H(z)e(t). \quad (2)$$

Here, $e(t) \in \mathbb{R}$ is white noise with variance σ_e^2, and $G_0(z)$ and $H(z)$ are stable, discrete-time transfer functions. Furthermore, $H(z)$ is assumed to be monic and minimum-phase. The signal $r(t) \in \mathbb{R}$ is the excitation signal $r(t) \in \mathbb{R}$ that is used to identify the true system. Suppose this excitation signal is applied from $t = 1$ to $t = N$ and that the corresponding input and output data are collected: $Z_N = \{u(t), y(t) \mid t = 1,\dots,N\}$. This data set can be used to determine an estimate of θ_0 in a full order model structure $\mathscr{M} = \{G(z,\theta)\}$: $\hat{\theta}_N = \arg\min_\theta V_{id}(\theta)$, with $V_{id}(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta)$, where $\varepsilon(t,\theta) = H^{-1}(z)\left[y(t) - G(z,\theta)u(t)\right]$ is the prediction error.

With the assumption of a model structure containing the true system, the identified parameter vector $\hat{\theta}_N$ is asymptotically normally distributed around the true parameter vector θ_0. Hence, under certain conditions on the excitation signal (and the controller in closed loop), we have for $N \to \infty$ that $\sqrt{N}(\hat{\theta}_N - \theta_0) \to \mathscr{N}(0, P_\theta)$, with P_θ^{-1} a positive scalar given by [6]

$$P_\theta^{-1} = \frac{\sigma_e^2}{N} \bar{E}\left[(\psi(t,\theta)\mid_{\theta_0})^2\right], \quad (3)$$

that can be estimated from $\hat{\theta}_N$ and Z_N. In this equation, the expectation operator $\bar{E}$ is defined as $\bar{E} f(t) \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E f(t)$, where E is the usual expectation operator. Lastly, $\psi(t,\theta) \equiv -\frac{\partial \varepsilon(t,\theta)}{\partial \theta}$, where $\varepsilon(t,\theta)$ is the prediction error defined previously.

It can be shown that (3) admits a frequency domain expression given by [6]

$$P_\theta^{-1} = \frac{N}{2\pi\sigma_e^2} \int_{-\pi}^{\pi} |F_r(\omega,\theta_0)|^2 \Phi_r(\omega)\, d\omega + R_0, \quad (4)$$

where $R_0 = \frac{N}{2\pi}\int_{-\pi}^{\pi} |F_v(\omega,\theta_0)|^2\, d\omega$, and $F_r(\omega,\theta_0) = H^{-1}(e^{-i\omega})S(\omega,\theta_0)\Lambda_G(\omega,\theta_0)$, $F_v(\omega,\theta_0) = -C(e^{-i\omega})S(\omega,\theta_0)\Lambda_G(\omega,\theta_0)$. In these expressions, $S(\omega,\theta) = (1 + C(e^{-i\omega})G(e^{-i\omega},\theta))^{-1}$ is the sensitivity function of the closed-loop system, $\Lambda_G(\omega) = \frac{\partial G(e^{-i\omega},\theta)}{\partial\theta}\mid_{\theta_0}$, and $\Phi_r(\omega)$ is the power spectrum of the excitation signal $r(t)$ in (1). The spectrum is an even function in ω.

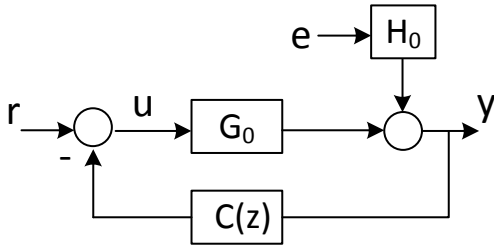

Fig. 1. Schematic overview of the true closed-loop system.

With these notations, the least-costly experiment design problem can be defined mathematically as

$$\min_{\Phi_r(\omega)} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{F}[\Phi_r(\omega),\omega]\, d\omega \quad (5)$$
$$\text{subject to } P_\theta^{-1} \geq R_{adm},$$
$$\text{and } \Phi_r(\omega) \geq 0\ \forall\omega,$$

with P_θ^{-1} as defined in (4), $\mathscr{F}[\Phi_r(\omega),\omega]$ a (possible nonlinear) objective function, and R_{adm} a scalar value calculated prior to the experiment, guaranteeing a certainty of the to-be-identified parameter that is large enough to enable robust control design. For more details on the computation of R_{adm}, see [4].

The solution to this problem, $\Phi_r^{opt}(\omega)$, is used to generate a time-domain excitation signal $r(t) = r^{opt}(t)$, which is used in the identification method explained previously. This completely defines the least-costly experiment.

The objective function $\mathscr{F}[\Phi_r(\omega),\omega]$ is, in literature, most often linear in $\Phi_r(\omega)$. Many functions fall in the class of $\mathscr{F}[\Phi_r(\omega),\omega] = \mathscr{L}(\omega)\Phi_r(\omega)$, where we assume that $\mathscr{L}(\omega) > 0\forall\omega \in [-\pi,\pi]$ and is sufficiently smooth. This is not a restrictive assumption, as $\mathscr{L}$ is a weighting function, and it therefore makes no sense for it to have negative values.

A popular choice of $\mathscr{L}(\omega)$ reads

$$\mathscr{L}(\omega) = \left(\alpha|G_0(e^{-i\omega})|^2 + \beta\right)|S(\omega,\theta_0)|^2, \quad (6)$$

which results in objective function consisting of the power of the perturbations induced by $r(t)$ on the input and the output. The parameters $\alpha,\beta \in \mathbb{R}$ are constant weightings [2].

Until now, we have considered the identification of θ_0 in a closed-loop setting. However, the identification can also be performed in an open-loop setting. The open-loop setting corresponds to the case where $C = 0$ and thus $u(t) = r(t)$. The excitation signal $r(t)$ is thus directly applied to the input of the system 1. In the open-loop setting it thus follows that $R_0 = 0$ and the sensitivity $S = 1$ in (4). The popular weighting

Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.

function for the open-loop case is thus

$$\mathscr{L}(\omega) = \left(\alpha |G_0(e^{-i\omega})|^2 + \beta\right). \tag{7}$$

In the Introduction, we mentioned that the optimization problem for objective functions linear in $\Phi_r(\omega)$ can be cast into a convex problem, which can be solved numerically. However, the somewhat unsatisfactory aspect in this procedure is the inability to obtain a good understanding of the resulting excitation spectrum.

In the following section, we adopt an analytical approach that provides insight into the solution of (5). For the sake of brevity, we will from now on omit the argument θ_0 in all terms in (4).

III. ANALYTICAL SOLUTION TO THE LEAST-COSTLY PROBLEM

In Section III-A, we address objective functions in (5) that are linear in $\Phi_r(\omega)$. Then, in Section III-B, we show how to solve the optimization problem for objective functions nonlinear in $\Phi_r(\omega)$. We will consider functions $\mathscr{F}[\Phi_r(\omega),\omega]$ that are even in ω, but note that all results can easily be extended to non-even functions. In such a case, one has to replace $\mathscr{F}[\Phi_r(\omega),\omega]$ with $\frac{1}{2}\mathscr{F}_T[\Phi_r(\omega),\omega] = \frac{1}{2}\left(\mathscr{F}[\Phi_r(\omega),-\omega]+\mathscr{F}[\Phi_r(\omega),\omega]\right)$ in the equations below.

In the following, we have made use of the fact that $|F_r(\omega)|^2$, $|F_v(\omega)|^2$, and $\Phi_r(\omega)$ are even functions in ω. Lastly, we remark that the solution is given for the closed-loop case, but that the open-loop case can be derived from it as a particular case.

A. Objective function linear in $\Phi_r(\omega)$

We consider (5) for $\mathscr{F}[\Phi_r(\omega),\omega] = \mathscr{L}(\omega)\Phi_r(\omega)$, where $\mathscr{L}(\omega) > 0 \forall \omega$ is an even function. The optimization problem thus reads:

$$\min_{\Phi_r(\omega)} \frac{1}{\pi}\int_0^\pi \mathscr{L}(\omega)\Phi_r(\omega)d\omega \tag{8}$$

subject to

$$\frac{1}{\pi}\int_0^\pi |F_r(\omega)|^2\Phi_r(\omega)d\omega \geq K \tag{9}$$

$$\text{and } \Phi_r(\omega) \geq 0 \,\forall\omega, \tag{10}$$

where $K = \frac{\sigma_e^2}{N}(R_{adm}-R_0)$. We assume that $R_{adm} > R_0$. Otherwise, the contribution of the noise to P_θ^{-1} is sufficient to satisfy the first constraint in (5) and the optimal excitation signal is simply $r(t) = 0, \forall t$.

We are now ready to derive the solution to the above problem, and show that this solution is unique.

Proposition 1: Consider the optimization problem (8)-(10). The spectrum solving this problem reads:

$$\Phi_r^{opt}(\omega) = \frac{\pi\sigma_e^2(R_{adm}-R_0)}{N|F_r(\omega^\dagger)|^2}\sum_{k=\{-1,1\}}\delta(\omega - k\omega^\dagger) \tag{11}$$

with

$$\omega^\dagger = \arg\max_{\omega\in[0,\pi]} \frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)}. \tag{12}$$

The excitation signal in the time-domain is given by

$$r^{opt}(t) = \frac{\sigma_e\sqrt{2}}{|F_r(\omega^\dagger)|}\sqrt{\frac{(R_{adm}-R_0)}{N}}\cos(\omega^\dagger t + \phi),$$

where ϕ is an arbitrary phase.

Proof: First, we omit constraint (10). We thus need to verify after our calculations that our solution satisfies this constraint. With $\omega^\dagger$ defined in (12), for all eventual solutions Φ_r to the optimization problem, the following inequality should hold:

$$K \leq \frac{1}{\pi}\int_0^\pi \frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)}\mathscr{L}(\omega)\Phi_r(\omega)\,d\omega \leq \frac{1}{\pi}\frac{|F_r(\omega^\dagger)|^2}{\mathscr{L}(\omega^\dagger)}\int_0^\pi \mathscr{L}(\omega)\Phi_r(\omega)\,d\omega, \tag{13}$$

where the first inequality is the constraint (9), while the latter is derived using (12). The last term is thus greater than or equal to K. Multiplying the inequality equation formed by the first and latter terms of (13) with $\frac{\mathscr{L}(\omega^\dagger)}{|F_r(\omega^\dagger)|^2}$ shows that the minimum of (8) subject to (9) is equal to $\frac{\mathscr{L}(\omega^\dagger)}{|F_r(\omega^\dagger)|^2}K$. With K defined in (9), it is obvious that this minimal cost:

$$\int_0^\pi \mathscr{L}(\omega)\Phi_r(\omega)d\omega = \frac{\mathscr{L}(\omega^\dagger)}{|F_r(\omega^\dagger)|^2}K \tag{14}$$

is obtained for the spectrum $\Phi_r(\omega) = \Phi_r^{opt}(\omega)$ given in (11). This solution is even by construction, and satisfies the constraint (10), that we had disregarded previously. ■

Proposition 2: Solution (11) is the unique minimizer of (8)-(10).

Proof: Suppose another solution, $\hat{\Phi}_r^{opt}(\omega) = \Phi_r^{opt}(\omega) + \eta(\omega)$ exists. Then this solution should be even ($\hat{\Phi}_r^{opt}(\omega) = \hat{\Phi}_r^{opt}(-\omega)$) and positive ($\hat{\Phi}_r^{opt}(\omega) \geq 0 \forall\omega$). This in turn means that $\eta(\omega) = \eta(-\omega)$, and that η must be positive everywhere except at $\omega = \omega^\dagger$, where it may be negative.

Substitution of $\Phi_r(\omega) = \hat{\Phi}_r^{opt}(\omega)$ into (14) yields

$$\frac{1}{\pi}\int_0^\pi \mathscr{L}(\omega)(\Phi_r^{opt}(\omega)+\eta(\omega))d\omega = \frac{\mathscr{L}_T(\omega^\dagger)}{|F_r(\omega^\dagger)|^2}K + \frac{1}{\pi}\int_0^\pi \mathscr{L}(\omega)\eta(\omega)\,d\omega.$$

The minimum of (8) is given by (14). Hence, in order to obtain the same minimum, the condition

$$\int_0^\pi \mathscr{L}(\omega)\eta(\omega)\,d\omega = 0 \tag{15}$$

must hold. The function $\eta(\omega)$ may only be negative at the point $\omega = \omega^\dagger$. Hence, this point must be given infinite weight in order to generate a negative contribution to the above integral. Therefore, the only possible solution for $\eta(\omega)$ is of the form

$$\eta(\omega) = f(\omega) + B\delta(\omega - \omega^\dagger), \tag{16}$$

where $B < 0$ is a constant, and $f(\omega) \geq 0 \forall\omega$ is an even, well-behaved function. Indeed, this solution satisfies all the conditions on $\eta(\omega)$ stated above. Both B and $f(\omega)$ are yet to be determined. Furthermore, we do not allow $f(\omega) =$

Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.

$-B\delta(\omega-\omega^\dagger)$, as this will result in the trivial solution $\eta(\omega)=0\,\forall\omega$. Substitution of (16) into (15) fixes B, which is given by

$$B=-\frac{1}{\mathscr{L}(\omega^\dagger)}\int_0^\pi \mathscr{L}(\omega)f(\omega)\,d\omega. \tag{17}$$

Condition (15) is now satisfied. The remaining freedom, $f(\omega)$, is to be used such that the constraint (9) with the potential solution $\hat{\Phi}_r^{opt}$ is satisfied. Substitution of $\Phi_r(\omega)=\hat{\Phi}_r^{opt}=\Phi_r^{opt}(\omega)+\eta(\omega)$ in (9), and using (13), (14), and (17) give

$$\begin{aligned}&\frac{1}{\pi}\int_0^\pi |F_r(\omega)|^2\hat{\Phi}_r^{opt}(\omega)d\omega=\\ &K+\frac{1}{\pi}\int_0^\pi |F_r(\omega)|^2 f(\omega)\,d\omega+\frac{B}{\pi}|F_r(\omega^\dagger)|^2=\\ &K+\frac{1}{\pi}\int_0^\pi \left[\frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)}-\frac{|F_r(\omega^\dagger)|^2}{\mathscr{L}(\omega^\dagger)}\right]\mathscr{L}(\omega)f(\omega)\,d\omega.\end{aligned}$$

By virtue of (12), $\frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)}\le\frac{|F_r(\omega^\dagger)|^2}{\mathscr{L}(\omega^\dagger)}$ for all $\omega\in[0,\pi]$. It thus follows that the above equation is less than K for any nontrivial $f(\omega)$, while they may be equal for $f(\omega)=0\,\forall\omega$, but this results in $B=0$, such that $\eta(\omega)=0\,\forall\omega$. This is a trivial solution for which $\hat{\Phi}_r^{opt}=\Phi_r^{opt}$.

Thus, we have shown that we can not obtain the minimum (8) while also ensuring that the constraint (9) is satisfied by any solution other than Φ_r^{opt}.

Hence, we find that solution (11) is unique. ■

B. Objective function nonlinear in $\Phi_r(\omega)$

As shown in the proof of Proposition 1, with the optimal power spectrum Φ_r^{opt}, the inverse P_θ^{-1} of the variance of the identified parameter vector is equal to R_{adm}. In other words, the constraint (9) is equal to K. This is due to the fact that all terms in (8)-(9) are positive, such that an increase on the left hand side of the constraint (9) will result in a larger value of the objective function. The same argument holds for positive, nonlinear objective functions $\mathscr{F}[\Phi_r(\omega),\omega]$, which we will consider to be even. Positive but non-even $\mathscr{F}$ can be treated as well, see Section II. In this section, we address the question: What will be the optimal spectrum when the objective function is nonlinear in Φ_r? This question can not be addressed with numerical convex algorithms that explicitly require linearity.

The optimization problem (5) for nonlinear objective functions is thus to minimize

$$F[\Phi_r(\omega)]=\frac{1}{\pi}\int_0^\pi \mathscr{F}[\Phi_r(\omega),\omega]\,d\omega \tag{18}$$

subject to

$$G[\Phi_r(\omega)]=\frac{1}{\pi}\int_0^\pi |F_r(\omega)|^2\Phi_r(\omega)\,d\omega=K. \tag{19}$$

As in the linear case, we have disregarded the constraint $\Phi_r(\omega)\ge 0\,\forall\omega$. This is only permitted when the solution to the above problem is identical to the solution of the original problem (c.f. (5)), i.e., the solution is positive on the entire domain.

The problem we need to solve is classical: find the path of integration along which the integral (18) has its minimum subject to the constraint (19). This type of problem can be solved with variational calculus and the method of Lagrange multipliers. For details, see [3].

Using these methods, we will now show with a simple example that the solution Φ_r^{opt} for nonlinear objective functions is no longer a sinusoid.

Example Let $\mathscr{F}[\Phi_r(\omega),\omega]=\mathscr{L}(\omega)\Phi_r^2(\omega)$, where $\mathscr{L}(\omega)>0\,\forall\omega$. It follows from variational calculus that

$$\Phi_r^{opt}(\omega)=\frac{2\pi K|F_r(\omega)|^2\mathscr{L}^{-1}(\omega)}{\int_0^\pi \mathscr{L}^{-1}(\omega')|F_r(\omega')|^4\,d\omega'}.$$

The solution $\Phi_r^{opt}(\omega)$ is continuous, positive, and even (by construction).

It can be shown that any (well-behaved) nonlinear objective function will generate a continuous spectrum in the interval $\omega\in[0,\pi]$. Hence, on leaving the linear domain, the solution can not be a sinusoid (or a finite sum of sinusoids).

IV. INTERPRETATION

We now return to the case of linear objective functions. In Section III, we saw that the unique solution to (8) is given by a sinusoid with frequency $\omega^\dagger$ (c.f. (12)) and an amplitude that is determined, besides other things, by the experiment length N, the sensitivity function, and the noise variance.

We observe from (12) that the optimal frequency $\omega=\omega^\dagger$ is located at the frequency for which the ratio $\frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)}$ is maximal. This is an intuitively appealing result.

For the sake of argument, let $\mathscr{L}(\omega)$ be a constant, say one. This means that the power of $r(t)$ at each frequency has equal weight in the objective function (8). Then the optimal frequency is where $|F_r(\omega)|^2$ attains its global maximum. Inspection of (9) indeed shows that this term should be maximized in order to minimize $\int_0^\pi\Phi_r(\omega)\,d\omega$ (or simply $\Phi_r(\omega)\,\forall\omega$, as $\Phi_r(\omega)\ge 0\,\forall\omega$). This observation has also been mentioned in [6] and observed numerically in [1].

When the weighting function $\mathscr{L}(\omega)$ is not a constant, not all terms $\Phi_r(\omega)$ are weighted equally in the objective function. However, by rewriting the optimization problem in the variable $\Xi(\omega)=\mathscr{L}(\omega)\Phi_r(\omega)$, the above reasoning may again be followed, where one now needs to maximize the ratio $|F_r(\omega)|^2/\mathscr{L}(\omega)$ (this follows trivially from the change of variables) in order to minimize the objective function $\int_0^\pi\Xi(\omega)\,d\omega$. Note that in this variable, the objective function again has a constant weighting function.

Let us consider transfer functions for which the unknown parameter θ_0 may now occur only *once*: in either the denominator or numerator. Let these functions be given by $G_1(z,\theta_0)=\frac{B(z)}{1+A(z,\theta_0)}$ and $G_2(z,\theta_0)=\frac{B(z,\theta_0)}{1+A(z)}$, where A, B are polynomials in z^{-1}. Note that this is a restriction from the general case we have considered so far.

Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.

For convenience, we recall the optimal excitation frequency:

$$\omega^\dagger = \arg\max_{\omega\in[0,\pi]} \frac{|F_r(\omega)|^2}{\mathscr{L}(\omega)} = \arg\max_{\omega\in[0,\pi]} \frac{|\Lambda_G(\omega)|^2|S(\omega)|^2}{|H(e^{-i\omega})|^2\mathscr{L}(\omega)}. \quad (20)$$

A first question that arises is: Will the optimal frequency depend on the location of θ_0 in $G(z,\theta_0)$? First, observe from the expression of $\omega^\dagger$ that only the term $|\Lambda_G(\omega)|^2$ could influence the optimal frequency location-wise, while all the others remain the same. We then need to distinguish two classes of transfer functions: (i) $G_1(z,\theta_0)$, and (ii) $G_2(z,\theta_0)$. For case (i), we find that $|\Lambda_G(\omega)|^2 = \frac{|B(e^{-i\omega})|^2}{|1+A(e^{-i\omega})|^4}$. This means that $\omega^\dagger$ depends on both the location of the poles and the zeros of $G_1(z,\theta_0)$, but is independent of the location of θ_0. For case (ii), $|\Lambda_G(\omega)|^2 = \frac{1}{|1+A(e^{-i\omega})|^2}$, showing that $\omega^\dagger$ only depends on the location of the poles of $G_2(z,\theta_0)$ but is invariant with respect to the position of θ_0 in the numerator. Hence, there are only two optimal frequencies for a uni-parametric model $G(z,\theta_0)$, of which the unknown θ_0 appears at only one location: one for identifying a parameter in the numerator and one for the denominator.

When the parameter appears in the more than one location in the numerator or denominator, the optimal frequency will depend on its locations.

We return to the general case of transfer functions $G(z,\theta_0)$, of which all parameters are known except θ_0, which may again appear at multiple locations in $A(z,\theta_0)$ and $B(z,\theta_0)$. We will now compare the least-costly identification experiment problem in the open-loop and the closed-loop settings. We recall that the open-loop setting is characterized by $R_0 = 0$, $S(\omega) = 1\,\forall\omega$ and $F_r(\omega) = \Lambda_G(\omega)/H(e^{-i\omega})$.

The cost of the optimal spectrum (11) is given by:

$$\begin{aligned} J_{opt} &= \frac{1}{\pi}\int_0^\pi \mathscr{L}(\omega)\Phi_r^{opt}(\omega)\,d\omega \\ &= \frac{\sigma_e^2}{N|F_r(\omega^\dagger)|^2}(R_{adm}-R_0)\mathscr{L}(\omega^\dagger). \end{aligned} \quad (21)$$

The ratio of optimal cost for the open- and closed-loop experiment, $f_{O/C} = \frac{J_{opt}^{OL}}{J_{opt}^{CL}}$, can be derived from the above equation, and reads

$$f_{O/C} = \frac{\mathscr{L}_O(\omega_O^\dagger)}{\mathscr{L}_C(\omega_C^\dagger)} \frac{|H(e^{i\omega_O^\dagger})|^2}{|H(e^{i\omega_C^\dagger})|^2} \frac{|\Lambda_G(\omega_C^\dagger)|^2}{|\Lambda_G(\omega_O^\dagger)|^2} \frac{|S_C(\omega_C^\dagger)|^2}{1-R_0/R_{adm}}, \quad (22)$$

where the subscripts O and C refer to the open- and closed-loop settings. For any family of $\mathscr{L}$, (22) shows that there may be cases in which an open-loop experiment is cheaper than a closed-loop one.

From now on, we restrict attention to the case where the cost of the identification is factorized as $\mathscr{L}(\omega) = \mathscr{L}_{in}(\omega)|S(\omega)|^2$, where $\mathscr{L}_{in}$ is identical for the open- and closed-loop cases, i.e., controller-independent. This family of functions includes the popular choice (6)-(7). Consequently, this factorization results in a frequency of the optimal excitation signal that is identical for the open-loop and in the closed-loop settings, and is given by (c.f. (20)):

$$\omega^\dagger = \arg\max_{\omega\in[0,\pi]} \frac{|\Lambda_G(\omega)|^2}{|H(e^{-i\omega})|^2\mathscr{L}_{in}(\omega)}. \quad (23)$$

The ratio of the open- and closed-loop experiment costs (22) in this case then reduces to $f_{O/C} = \frac{1}{1-R_0/R_{adm}}$. This equation clearly reveals the advantage of the closed-loop setting with respect to the open-loop setting for this family of $\mathscr{L}$. This benefit of closed-loop identification is due to the term R_0 i.e. the accuracy that is due to the noise. Observe that for $R_{adm} \gg R_0$ this fraction approach its limit $f_{O/C} = 1$, in which case both experiments yield equal costs.

This observation is intricately related to the amplitude of the excitation signal $r(t)$. Indeed, from (11), (13), and (21), it follows that the cost is proportional to the square root of the amplitude of $r(t)$. The amplitude of $r(t)$ in the closed-loop case is thus always smaller than the one in the open-loop case. What does this mean for the power of the input and output, i.e., the overall level of perturbation? If the system is already operated in closed-loop and if the controller effectively reduces the effects of the noise on the output, i.e., $||SH_0||_2 < 1$, it is always beneficial to perform the identification in closed loop. The power of the output $y(t) = G_0Sr(t) + H_0Se(t)$ and input $u(t) = r(t) - C(z)y(t)$ for $r(t) = r^{opt}(t)$ in the closed-loop case (c.f. (11)):

$$\begin{aligned} \mathscr{P}_y &= |G_0(e^{-i\omega^\dagger})|^2 \frac{\sigma_e^2\pi|H(e^{-i\omega^\dagger})|^2}{N|\Lambda_G(\omega^\dagger)|^2}(R_{adm}-R_0) \\ &+ |H(e^{-i\omega^\dagger})|^2|S(\omega^\dagger)|^2\sigma_e^2, \\ \mathscr{P}_u &= \frac{\pi\sigma_e^2|H(e^{-i\omega^\dagger})|^2}{N|\Lambda_G(\omega^\dagger)|^2}(R_{adm}-R_0) \\ &+ |S(e^{-i\omega^\dagger})|^2|H(e^{-i\omega^\dagger})|^2|C(e^{-i\omega^\dagger})|^2\sigma_e^2. \end{aligned} \quad (24)$$

The first equation trivially shows that if $||SH_0||_2 < 1$ holds, the power of $y(t)$ in closed-loop will always be smaller than in the open-loop case (for which $\omega^\dagger$ is identical, but $R_0 = 0$). The same conclusion holds for the input $u(t)$, even without the above assumption. Indeed, inspection of R_0, see (4), shows that the term $\frac{\sigma_e^2\pi|H(e^{-i\omega^\dagger})|^2}{N|\Lambda_G(\omega^\dagger)|^2}R_0$ is always larger than the last term of $\mathscr{P}_u$.

V. A CASE STUDY

The aim of this section is to verify our theoretical findings of Section III.

To this end, we consider the true system $G_0(e^{-i\omega}) = \frac{a_1(1-\theta_0)e^{-i\omega}}{1-\theta_0 e^{-i\omega}}$, and $H(e^{-i\omega}) = 1$, where $a_1 = 10$ is the steady-state gain, $\sigma_e^2 = 4.0$ is the variance of the noise $e(t)$, and $\theta_0 = 0.45$ is the unknown parameter that we need to identify with a least-costly experiment. We first consider the closed-loop setting with a PI controller $C(z)$ defined by the constants $K_p = 0.00853$ and $K_i = 0.0171$. With this controller $R_0 = 3.0826$.

The optimization problem we consider is (8), with $\mathscr{L}(\omega)$ given by (6). The parameters defining the optimization problem are: $N = 200$, $\alpha = 0.1$, $\beta = 0.7$, and $R_{adm} = 1000$.

We first consider the theoretical predictions. The optimal excitation is given by (13) and is here equal to

Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.

$r^{opt}(t) = 0.5033\cos(\omega^\dagger t + \phi)$ with an arbitrary phase shift ϕ and with $\omega^\dagger = 1.8903$ rad/s. The frequency $\omega^\dagger$ can be determined by inspection of the frequency response of $|F_r(\omega)|^2/L(\omega)$ or, alternatively, via straightforward but tedious algebraic manipulations leading to the expression $\omega^\dagger = \arccos\left[1 - \frac{(1-\theta_0)^2}{2\beta\theta_0}\sqrt{\beta(\beta+\alpha a_1^2)}\right]$. The cost of the identification corresponding to this identification is equal to $J_{opt} = 0.4580$.

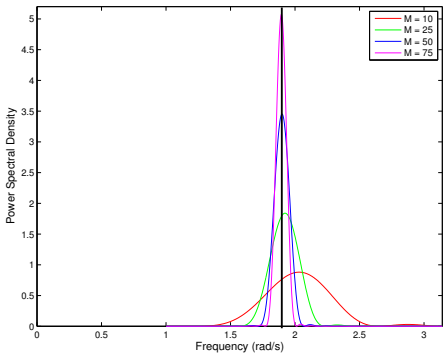

Fig. 2. Optimal excitation spectra for $M = 10$ (red), $M = 25$ (green), $M = 50$ (blue), and $M = 75$ (purple). The black vertical line is the analytical optimal spectrum. The parameter values for the simulations are $\theta_0 = 0.45$, $N = 200$, $a_1 = 10$, $\sigma_e^2 = 4.0$, and $R_{adm} = 1000$. Observe that when M is increased, the numerical spectra converge to the analytical solution.

We will compare this optimal excitation with the excitation that we obtain via convex optimization. As already mentioned, the least costly problem is usually solved numerically using convex optimization. Although other parameterizations are possible, we use the following class of power spectra in this convex optimization [2]: $\Phi_r(\omega) = \sum_{k=-M}^{M} c_k e^{-i\omega k}$, where $c_{-k} = c_k$ are to-be-determined coefficients and M the order. Note that, for finite M, this class does not contain all possible spectra and, in particular, does not contain the optimal spectra (11).

We have solved the least-costly problem in this example using this numerical routine for different values of M, i.e., $M = \{10, 25, 50, 75\}$. The obtained spectra are depicted in Figure 2 and we observe that, for small values of M, the obtained spectra are relatively far from the optimal one which is a Dirac pulse at $\omega^\dagger = 1.8903$ rad/s. This discrepancy is also observed in the optimal identification cost that are respectively given by $J_{opt} = \{0.4601, 0.4584, 0.4581, 0.4580\}$ for these four values of M. We see that only for the large values of M the cost converges to the optimal one, i.e., the one obtained with the sinusoid excitation.

We now compare the closed-loop setting and the open-loop setting. Recall that, for open-loop identification, the identification cost is defined by (21). The optimal excitation signal is here $r_{OL}^{opt}(t) = 0.5273\cos(\omega^\dagger t + \phi)$ with the same frequency $\omega^\dagger$ as in the closed-loop case, but with another amplitude. Since the controller reduces the effects of the disturbance on the output and given the relation (24), the optimal open-loop experiment leads to a signal y with a higher variance that the optimal closed-loop experiment, c.f. (24). The differences are not striking because $R_{adm} >> R_0$, as observed from (22). However, if R_{adm} is chosen equal to $1.1R_0$, the differences between the open-loop and closed-loop settings are much larger. Note that the relation (24) and $f_{O/C} = \frac{1}{1-R_0/R_{adm}}$ only hold since (6) is of the form $\mathscr{L}(\omega) = \mathscr{L}_{in}(\omega)|S(\omega)|^2$. To illustrate this, as a last experiment, we change the objective/cost function to $\mathscr{L}(\omega) = 1 \forall \omega$ (i.e. the experiment cost is thus now solely the power of the excitation signal). All parameter values are kept as defined above ($R_{adm} = 1000$). The fraction of costs (22) is now $f_{O/C} = 0.843$, indicating that an open-loop experiment is cheaper than a closed-loop one.

VI. CONCLUSIONS

We have calculated the excitation signal that is required to identify a uni-parametric linear system with the least-costly framework. This enabled us to address several fundamental questions about the shape of optimal signal and the cost of the identification experiment.

We have shown, among other things, that for an objective function linear in the power spectrum, the solution is a sinusoid with a unique frequency and amplitude, and what determines these. Using this solution, it is shown that for $\mathscr{L}(\omega) \propto |S(\omega)|^2$, the optimal excitation frequency is identical for a system being in an open or a closed loop, and that for this case a closed-loop experiment is always cheaper than an open-loop one. We also proved that the optimal frequency is only different, location-wise, for the to-be-identified parameter residing in solely one position in the numerator or denominator. This is, however, not true if the parameter appears at multiple positions.

Lastly, a case study illustrates some of the theoretical insights.

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Preprint submitted to 53rd IEEE Conference on Decision and Control.
Received March 18, 2014.