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$\mathcal{H}_\infty$ performance analysis and distributed controller synthesis for interconnected linear systems from noisy input-state data

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Abstract— **The increase in available data and complexity of dynamical systems has sparked the research on data-based system performance analysis and controller design. In this paper, we extend a recent data-based approach for guaranteed performance analysis to distributed analysis of interconnected linear systems. We present a new set of sufficient LMI conditions based on noisy input-state data that guarantees $\mathcal{H}_\infty$ performance and has a structure that is applicable to distributed controller synthesis from data. Sufficient LMI conditions based on noisy data are provided for the existence of a dynamic distributed controller that achieves $\mathcal{H}_\infty$ performance. The presented approach enables scalable analysis and control of large-scale interconnected systems from noisy input-state data.**

I. INTRODUCTION

Data is becoming increasingly relevant for the analysis and control of dynamical systems. The rise in complexity of systems implies that the well-known model-based approaches can become unsuitable in applications for which the mathematical modelling is tedious. Especially for interconnected systems, such as smart grids, smart buildings or industrial processes, models are not readily available and the spatial distribution or dimensionality complicates first-principles modelling. On the other hand, data is available with increased ease. Data can be used either indirectly by performing system identification with model-based analysis and control, or directly via data-based system analysis and controller synthesis.

Several methods have been developed for data-based system analysis and controller synthesis, we refer to [1] for a survey on data-based control. Some methods rely on the reference model paradigm, such as virtual reference feedback tuning [2] and optimal controller identification [3]. Extensions for interconnected systems to data-based distributed controller synthesis include distributed virtual reference feedback tuning in the noiseless [4], and noisy [5] case.

A recent trend in data-based system analysis and control originates from Willems' fundamental lemma [6]. Applications include data-based predictive control [7], [8], the data-based parameterization of stabilizing controllers [9] and robust data-based state-feedback design with noisy data [10]. The data-based verification of dissipativity properties was considered in [11], [12], which allows to determine system measures such as the $\mathcal{H}_\infty$ norm or passivity properties from data corrupted by a noise signal satisfying quadratic bounds. A similar noise description was considered in [13], which extends the data-based controller design results in [14] to the noisy case. The data-based conditions in [13] are necessary and sufficient for stabilizing state feedback synthesis, including $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance specifications.

In this paper, the data-based $\mathcal{H}_\infty$ performance analysis and *distributed* controller synthesis problem for *interconnected* systems is considered. We extend the data-based framework for parameterizing an unknown system, considered in the two distinct papers [12] and [13], to the situation of interconnected systems. The analysis in this paper is enabled by considering a dual parameterization of the set $\Sigma_{\mathcal{D}}$: the set of systems that are compatible with input-state data $\mathcal{D}$ for unmeasured noise trajectories in a set $\mathcal{W}$ that captures quadratic bounds on the noise sequence. A feature of the dual parameterization is the applicability of standard (primal) conditions for unstructured [15] and structured [16] robust performance analysis. For an interconnected system, we consider sets $\Sigma^i_{\mathcal{D}}$ of subsystems that are compatible with the local input-state and neighbors' state data, given prior knowledge on the noise signals confined to a set $\mathcal{W}_i$. We develop sufficient data-based conditions for $\mathcal{H}_\infty$ performance analysis and for the existence of a dynamic distributed controller that achieves a given $\mathcal{H}_\infty$ performance level.

A feature of our results is that no model of the interconnected system is identified from the data. The identification of interconnected systems is considered in the field of network identification, which provides structured and consistent methods for identification [17]. If an identified model is only used for controller synthesis, however, it is arguably more efficient to consider data-based synthesis conditions directly. Additionally, with our data-based method, stability and performance guarantees for the closed-loop interconnected system come with a finite number of data points. Comparatively, system-identification methods come with consistency results asymptotic in the number of data, but do not provide guarantees for finite data.

Basic nomenclature

The integers are denoted by $\mathbb{Z}$. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \dots, b-1, b\}$. Let $I_n \in \mathbb{R}^{n\times n}$, or simply I, denote the identity matrix and $\mathbf{1}_n \in \mathbb{R}^n$, or simply $\mathbf{1}$, denote the column vector of all ones. For a subset $A \subset \mathbb{Z}$, the vertical, respectively horizontal, stacking of matrices X_a, $a \in \mathcal{A}$ is denoted $\mathrm{col}_{a\in\mathcal{A}}\, X_a$, respectively $\mathrm{row}_{a\in\mathcal{A}}\, X_a$. The kernel of a matrix A is denoted $\ker A$ and

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a matrix $A_\perp$ denotes a basis matrix of $\ker A$. For a real symmetric matrix X, $X > 0$ ($X \geq 0$) denotes that X is positive (semi-) definite. Matrices that can be inferred from symmetry are denoted by $(\star)$.

II. Preliminaries

In this paper, we consider interconnected systems composed of L linear time-invariant systems of the form

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + \sum_{j\in\mathcal{N}_i} A_{ij} x_j(k) + B_i u_i(k) + w_i(k), \\ y_i(k) &= C_i x_i(k) + D_i u_i(k) \quad \text{for} \quad i = 1, \ldots, L, \quad (1) \end{aligned}$$

where $x_i \in \mathbb{R}^{n_i}$ denotes the state, $u_i \in \mathbb{R}^{m_i}$ the input and $w_i \in \mathbb{R}^{n_i}$ is a noise signal. The set $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$ denotes the neighbours of system i, where $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ denote the set of vertices and the set of non-oriented edges defining the connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Let there exist a true interconnected system defined by the matrices A_i^0, A_{ij}^0 and B_i^0, $(i,j) \in \mathcal{E}$, generating the input-state data $\{(u_i(t), x_i(t)),\, t = 0, \ldots, N\}$ for $i \in \mathcal{V}$. This data is collected in the matrices

$$X_i := [x_i(0) \ \cdots \ x_i(N)], \ U_i^- := [u_i(0) \ \cdots \ u_i(N-1)].$$

By defining the matrices

$$\begin{aligned} &X_i^+ := [x_i(1) \ \cdots \ x_i(N)], \ X_i^- := [x_i(0) \ \cdots \ x_i(N-1)], \\ &W_i^- := [w_i(0) \ \cdots \ w_i(N-1)], \end{aligned}$$

we obtain the following data equation for each $i \in \mathcal{V}$:

$$X_i^+ = A_i^0 X_i^- + \sum_{j\in\mathcal{N}_i} A_{ij}^0 X_j^- + B_i^0 U_i^- + W_i^-. \quad (2)$$

Consider the stacked input, state and noise variables $u := \mathrm{col}(u_1, \ldots, u_L)$, $x := \mathrm{col}(x_1, \ldots, x_L)$ and $w := \mathrm{col}(w_1, \ldots, w_L)$. Then the interconnected system (1) is compactly described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k), \quad (3) \\ y(k) &= Cx(k) + Du(k), \end{aligned}$$

with straightforward definitions for A, B, C and D. The corresponding data equation is

$$X_+ = A_0 X_- + B_0 U_- + W_-,$$

with the data matrices defined for system (3) as was done for each subsystem. The transfer matrix from u to y of (3) is $G(q) := C(qI-A)^{-1}B + D$ and the $\mathcal{H}_\infty$ norm of G is denoted $\|G\|_{\mathcal{H}_\infty}$. For $\gamma > 0$, we say that the interconnected system achieves $\mathcal{H}_\infty$ performance γ if $\|G\|_{\mathcal{H}_\infty} < \gamma$.

III. Inferring system performance from noisy data

In this section, we consider the data-based dissipativity analysis for an unstructured system. We recall a parameterization from [12] and introduce a dual parameterization of systems that are compatible with input-state data. The dual parameterization allow us to (i) derive a dual result with respect to [12] for concluding dissipativity properties from data, and (ii) extend the data-based results to structured results for interconnected systems.

Consider the system

$$\begin{aligned} x(k+1) &= A_0 x(k) + B_0 u(k) + w(k), \quad (4) \\ y(k) &= Cx(k) + Du(k) \quad (5) \end{aligned}$$

with collected data

$$\begin{aligned} &X_+ := [x(1) \ \cdots \ x(N)], \ X_- := [x(0) \ \cdots \ x(N-1)], \\ &U_- := [u(0) \ \cdots \ u(N-1)], \end{aligned}$$

and noise sequence $W_- := [w(0) \ \cdots w(N-1)]$. We assume that the data (U_-, X) are known, while W_- is unknown, but

$$W_- \in \mathcal{W} := \left\{ W \,\middle|\, \begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix} \begin{bmatrix} W^\top \\ I \end{bmatrix} \geq 0 \right\},$$

with $Q_w < 0$ so that $\mathcal{W}$ is bounded. No assumptions on the statistics of w are made. This noise model can represent, e.g., an energy bound $W_- W_-^\top \leq R_w$ for $Q_w = -I$ and $S_w = 0$ or bounds on individual components $w(k)$ [13]. The square of sample cross-covariance bounds, as considered in [18] for parameter-bounding identification, can be captured by $\mathcal{W}$ with Q_w generally not strictly negative definite; this is a topic of current research. We assume that the data are informative in the sense that the matrix $\mathrm{col}(X_-, U_-)$ has full row rank.

Because the noise term is unknown, there exist multiple pairs of system matrices that are consistent with the data. The set of all pairs (A,B) that are consistent with the data is defined as

$$\Sigma_\mathcal{D} = \{(A,B) \,|\, X_+ = AX_- + BU_- + W \text{ for some } W \in \mathcal{W}\}.$$

We note that the true system $(A_0, B_0) \in \Sigma_\mathcal{D}$ by construction. Furthermore, in the *noiseless* case ($W_- = 0$), $\Sigma_\mathcal{D}$ reduces to the singleton $\{(A_0, B_0)\}$ if $\mathrm{col}(X_-, U_-)$ has full rank [14].

The following result from [13], cf. [12], provides a parametrization of the set $\Sigma_\mathcal{D}$.

Lemma III.1 (Parameterization $\Sigma_\mathcal{D}$) *It holds that*

$$\Sigma_\mathcal{D} = \{(A,B) \,|\, \begin{bmatrix} -A^\top \\ -B^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \bar{Q}_\mathcal{D} & \bar{S}_\mathcal{D} \\ \bar{S}_\mathcal{D}^\top & \bar{R}_\mathcal{D} \end{bmatrix} \begin{bmatrix} -A^\top \\ -B^\top \\ I \end{bmatrix} \geq 0\},$$

with $$\begin{bmatrix} \bar{Q}_\mathcal{D} & \bar{S}_\mathcal{D} \\ \bar{S}_\mathcal{D}^\top & \bar{R}_\mathcal{D} \end{bmatrix} = \begin{bmatrix} X_- & 0 \\ U_- & 0 \\ X_+ & I \end{bmatrix} \begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix} \begin{bmatrix} X_- & 0 \\ U_- & 0 \\ X_+ & I \end{bmatrix}^\top.$$

We now present a dual parameterization of $\Sigma_\mathcal{D}$.

Lemma III.2 (Dual parameterization $\Sigma_\mathcal{D}$) *Let the matrix*

$$\begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix}$$

be invertible. Then it holds that

$$\Sigma_\mathcal{D} = \{(A,B) \,|\, \begin{bmatrix} I & 0 \\ 0 & I \\ A & B \end{bmatrix}^\top \begin{bmatrix} Q_\mathcal{D} & S_\mathcal{D} \\ S_\mathcal{D}^\top & R_\mathcal{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ A & B \end{bmatrix} \leq 0\},$$

where $R_{\mathcal{D}} > 0$ with $\left[\begin{smallmatrix} Q_{\mathcal{D}} & S_{\mathcal{D}} \\ S_{\mathcal{D}}^\top & R_{\mathcal{D}} \end{smallmatrix}\right] := \left[\begin{smallmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{smallmatrix}\right]^{-1}$.

Proof: We refer to Appendix I in [19] for the proof. ■

Since any system that is consistent with the data is an element of $\Sigma_{\mathcal{D}}$, every such system admits a representation

$$x(k+1) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \quad \text{with } (A,B) \in \Sigma_{\mathcal{D}}.$$

As it was shown in [12], this uncertain system admits the following linear fractional transformation (LFT) representation

$$\begin{bmatrix} x(k+1) \\ y(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ C & D & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \\ l(k) \end{bmatrix}, \; l(k) = \begin{bmatrix} A & B \end{bmatrix} p(k),$$

with $(A,B) \in \Sigma_{\mathcal{D}}$.

Proposition III.1 (Dissipativity from data) *If there exist a P and α such that $P > 0$, $\alpha > 0$ and (6) hold, then*

$$\begin{bmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{bmatrix}^\top \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^\top & -R \end{array}\right] \begin{bmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{bmatrix} < 0 \quad (7)$$

holds for all $(A,B) \in \Sigma_{\mathcal{D}}$.

Proof: Let (6) hold and let $M = \begin{bmatrix} A & B \end{bmatrix}$. By Lemma III.2, it holds that for $\alpha > 0$,

$$\begin{bmatrix} I \\ M \end{bmatrix}^\top \begin{bmatrix} -\alpha Q_{\mathcal{D}} & -\alpha S_{\mathcal{D}} \\ -\alpha S_{\mathcal{D}}^\top & -\alpha R_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} \geq 0$$

for all $(A,B) \in \Sigma_{\mathcal{D}}$. Therefore, by the full block S-procedure [15], it follows that (7) holds. ■

Inequality (7) is the well known condition for dissipativity for a quadratic supply rate matrix $\Pi = -\left[\begin{smallmatrix} Q & S \\ S^\top & R \end{smallmatrix}\right]$. A special case of the supply rate matrix is $Q = \gamma^2 I$, $S = 0$ and $R = -I$ for $\gamma > 0$. For this specific case there exists a $P > 0$ so that (7) holds if and only if the channel $u \to y$ achieves $\mathcal{H}_\infty$ performance γ.

We have derived a dual parameterization of $\Sigma_{\mathcal{D}}$, which allows the application of standard robust control tools to the LFT representation. The parameterization from Lemma III.1, see also [12, Lemma 2], requires the application of the dualization lemma on the data-based LMI. A feature of the dual parameterization of $\Sigma_{\mathcal{D}}$ in Lemma III.2, is that robust analysis tools for interconnected systems can be applied *mutatis mutandis*, as we will show in the next section.

IV. Interconnected system analysis

Let us return to the interconnected system (1). We consider the data U_i^-, X_i and X_j, $j \in \mathcal{N}_i$, is available for each system i, while W_i^- is unknown. For each $i \in \mathcal{V}$, we assume

$$W_i^- \in \mathcal{W}_i = \left\{ W_i \,|\, \begin{bmatrix} W_i^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q_w^i & S_w^i \\ (S_w^i)^\top & R_w^i \end{bmatrix}}_{=:\Pi_w^i} \begin{bmatrix} W_i^\top \\ I \end{bmatrix} \geq 0 \right\},$$

with $Q_w^i < 0$. We assume that the data are informative enough in the sense that the matrix $\mathrm{col}(X_i^-, X_{\mathcal{N}_i}^-, U_i^-)$ has full row rank for each $i \in \mathcal{V}$.

For each subsystem, there exist multiple tuples $(A_i, A_{\mathcal{N}_i}, B_i)$ that are consistent with the data, i.e., that satisfy

$$X_i^+ = A_i X_i^- + \sum_{j \in \mathcal{N}_i} A_{ij} X_j^- + B_i U_i^- + B_i^w W_i \quad (8)$$

for some $W_i \in \mathcal{W}_i$. Here, we define $A_{\mathcal{N}_i} := \mathrm{row}_{j \in \mathcal{N}_i} A_{ij}$. Hence, for each $i \in \mathcal{V}$, the set of subsystems that are consistent with the data is

$$\Sigma_{\mathcal{D}}^i := \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (8) \text{ holds for some } W_i \in \mathcal{W}_i\}$$

We note that under the assumption that $W_i^- \in \mathcal{W}_i$, the true system matrices are in the set $\Sigma_{\mathcal{D}}^i$ by construction.

Lemma IV.1 (Parameterization $\Sigma_{\mathcal{D}}^i$) *It holds that*

$$\Sigma_{\mathcal{D}}^i = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \begin{bmatrix} \bar{Q}_{\mathcal{D}}^i & \bar{S}_{\mathcal{D}}^i \\ (\bar{S}_{\mathcal{D}}^i)^\top & \bar{R}_{\mathcal{D}}^i \end{bmatrix} \begin{bmatrix} -A_i^\top \\ -A_{\mathcal{N}_i}^\top \\ -B_i^\top \\ I \end{bmatrix} \geq 0\},$$

with $$\begin{bmatrix} \bar{Q}_{\mathcal{D}}^i & \bar{S}_{\mathcal{D}}^i \\ (\bar{S}_{\mathcal{D}}^i)^\top & \bar{R}_{\mathcal{D}}^i \end{bmatrix} := \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix} \Pi_i^w \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix}^\top.$$

Lemma IV.2 (Dual parameterization $\Sigma_{\mathcal{D}}^i$) *Let Π_i^w be invertible. It holds that $\Sigma_{\mathcal{D}}^i$ is equal to*

$$\{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \begin{bmatrix} Q_{\mathcal{D}}^i & S_{\mathcal{D}}^i \\ (S_{\mathcal{D}}^i)^\top & R_{\mathcal{D}}^i \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} \leq 0\},$$

where $R_{\mathcal{D}}^i > 0$ with $\left[\begin{smallmatrix} Q_{\mathcal{D}}^i & S_{\mathcal{D}}^i \\ (S_{\mathcal{D}}^i)^\top & R_{\mathcal{D}}^i \end{smallmatrix}\right] := \left[\begin{smallmatrix} \bar{Q}_{\mathcal{D}}^i & \bar{S}_{\mathcal{D}}^i \\ (\bar{S}_{\mathcal{D}}^i)^\top & \bar{R}_{\mathcal{D}}^i \end{smallmatrix}\right]^{-1}$.

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & 0 & I \\ C & 0 & D \end{bmatrix}^\top \left[\begin{array}{cc|cc|cc} -P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\alpha R_{\mathcal{D}} & -\alpha S_{\mathcal{D}}^\top & 0 & 0 \\ 0 & 0 & -\alpha S_{\mathcal{D}} & -\alpha Q_{\mathcal{D}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -Q & -S \\ 0 & 0 & 0 & 0 & -S^\top & -R \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & 0 & I \\ C & 0 & D \end{bmatrix} < 0 \quad (6)$$

The proofs for Lemma IV.1 and IV.2 follow an analogue reasoning as the proofs for Lemma III.1 and III.2, respectively, and are omitted for brevity.

We note that if any interconnected system with subsystems in $\Sigma_{\mathcal{D}}^i$, i.e., any interconnected system that is consistent with the data, has a certain property, then also the true interconnected system has this property. To show a property for all interconnected systems that are consistent with the data, we use the following LFT representation.

Every interconnected system that is consistent with the data can be described by subsystems $\Sigma_{\mathcal{D}}^i$, $i \in \mathcal{V}$,

$$\begin{bmatrix} x_i(k+1) \\ y_i(k) \\ p_i(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & I \\ C_i & 0 & D_i & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_i(k) \\ \operatorname{col}_{j \in \mathcal{N}_i} x_j(k) \\ u_i(k) \\ l_i(k) \end{bmatrix}$$

and $l_i(k) = \begin{bmatrix} A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} p_i(k)$, with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$.

This LFT representation for each subsystem allows us to apply robust analysis results for interconnected systems, to conclude $\mathcal{H}_\infty$ performance for all interconnected systems that are compatible with the data. Consider the matrices Z_i defined in Appendix II in [19].

Proposition IV.1 (Performance from structured data) *Let $Q_{\mathcal{D}}^i < 0$ and $\gamma > 0$. If there exist P_i, Z_i and α_i so that $P_i > 0$, $\alpha_i > 0$ and* (9) *holds for all $i \in \mathcal{V}$, then all interconnected systems with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$, $i \in \mathcal{V}$, achieve $\mathcal{H}_\infty$ performance γ.*

The proof follows by a similar argument as in Proposition III.1 and the application of [16, Theorem 1] to the LFT representation, and it is omitted due to space limits.

V. Distributed controller synthesis from data

So far we have considered the performance analysis of (interconnected) systems from data for the channel $u \to y$. We will now consider a distributed control problem for the interconnected system (1), where we take u_i and y_i as the control input and measured output respectively. Recall that we assume that input-state data is collected to determine $\Sigma_{\mathcal{D}}^i$ for each i. With the system matrices defined as $C_i = I$, $D_i = 0$, this implies only state-measurements are available for control. We note, however, that C_i is allowed to be chosen arbitrarily in this section and that $D_i = 0$; this implies that output measurements can be utilized for control. Future research will focus on extending the framework to the case when only input-output data is available for synthesis. The problem under consideration is to guarantee that the channel $w \to z$ achieves $\mathcal{H}_\infty$ performance $\gamma > 0$, with performance output

$$z_i = C_i^z x_i + \sum_{j \in \mathcal{N}_i} C_{ij}^z x_j + D_i^z u_i. \tag{10}$$

We consider a distributed controller that is an interconnected system with dynamic subsystems

$$\begin{bmatrix} \xi_i(k+1) \\ o_i(k) \\ u_i(k) \end{bmatrix} = \Theta_i \begin{bmatrix} \xi_i(k) \\ s_i(k) \\ y_i(k) \end{bmatrix}, \quad i = 1, \ldots, L, \tag{11}$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state of controller i and $o_i = \operatorname{col}_{j \in \mathcal{N}_i} o_{ij}$, $s_i = \operatorname{col}_{j \in \mathcal{N}_i} s_{ij}$ are interconnection variables satisfying $s_{ij} = o_{ji} \in \mathbb{R}^{n_{ij}}$ for $(i,j) \in \mathcal{E}$. By representing every interconnected system with performance output (10) and subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$ in LFT form, we obtain conditions on the data for the existence of a distributed controller by [16, Theorem 2].

Theorem V.1 (Distributed control from data) *Let Ψ_i and Φ_i be matrices that are a basis of $\ker \begin{bmatrix} C_i & 0 \end{bmatrix}$ and $\ker \begin{bmatrix} 0 & I & (D_i^z)^\top \end{bmatrix}$, respectively, and let $n_{ij} = 3n_i$. If there exist P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$, α_i such that $P_i > 0$, $\bar{P}_i > 0$, $\alpha_i > 0$,* (12)-(13) *hold (see next page) with $\beta_i = \alpha_i^{-1}$ and*

$$\begin{bmatrix} P_i & I \\ I & \bar{P}_i \end{bmatrix} \geq 0,$$

then there exist Θ_i, $i \in \mathcal{V}$, so that all closed-loop interconnected systems described by (1), (10) *and* (11) *with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$ achieve $\mathcal{H}_\infty$ performance γ.*

Remark V.1 The conditions in Proposition V.1 are sufficient for any $\alpha_i > 0$ are LMIs for fixed α_i. Conservatism can be reduced by, e.g., verifying feasibility of the LMIs on a discrete interval for α_i, $i \in \mathcal{V}$.

In particular, Theorem V.1 implies that the existence of a distributed controller for which the 'true' interconnected system achieves $\mathcal{H}_\infty$ performance, can be verified by checking a set of LMIs based on noisy input-state data. Suitable matrices

$$(\star)^\top \left[\begin{array}{cc|cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_i R_{\mathcal{D}}^i & -\alpha_i (S_{\mathcal{D}}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_i S_{\mathcal{D}}^i & -\alpha_i Q_{\mathcal{D}}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline \mathbf{1} \otimes I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \\ \hline 0 & 0 & 0 & I \\ C_i & 0 & 0 & D_i \end{array}\right] < 0 \tag{9}$$

P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$ are thus indirectly based on the data; these matrices can be used for the subsequent construction of the controller matrices Θ_i as described in [20], cf. [16]. We note that neither our existence conditions, nor the construction of Θ_i is based on the unknown matrices $(A_i, A_{\mathcal{N}_i}, B_i)$.

VI. Examples

A. Example 1: $\mathcal{H}_\infty$-norm analysis

Consider a system of the form (4) with $L = 3$,

$$A_0 = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.4 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} \quad \text{and} \quad B_0 = I.$$

We choose $y = x$ so that $C = I$ and $D = 0$. The input entries are drawn from a normal distribution with zero mean and unit variance. The noise $w(k)$ is drawn uniformly from the set $\{w \mid \|w\|_2 \leq \sigma\}$, where $\sigma > 0$ determines the noise level. Hence, considering the set $\mathcal{W}$ with $Q_w = -I$, $S_w = 0$ and $R_w = N\sigma^2 I$, we have that the noise satisfies $W_- \in \mathcal{W}$.

The aim is to find an upperbound on the $\mathcal{H}_\infty$ norm of the channel $u \to y$ using the noisy data (U_-, X) with $N = 50$ samples. The true $\mathcal{H}_\infty$ norm is $\gamma_0 = 2.8836$. We choose eleven noise levels σ in the interval $[0.04, 0.25]$ and generate one data set for each noise level. For each data set, we minimize γ subject to (6) with $Q = \gamma^2 I$, $S = 0$ and $R = -I$. The results are displayed in Figure 1 in blue. By Proposition III.1, the corresponding solutions satisfy (7), hence γ is an upperbound on the $\mathcal{H}_\infty$ norm for all systems in $\Sigma_\mathcal{D}$ and, therefore, for (A_0, B_0).

Next, we perform the analysis through Proposition IV.1 using the same data sets. It is clear that $W_i^- \in \mathcal{W}_i$ for each i with $Q_w^i = -I$, $S_w^i = 0$ and $R_w^i = N\sigma^2 I$. For each data set, we minimize γ subject to the LMIs (9) for $i = 1, 2, 3$. The resulting γ values provide a guaranteed upper bound on the $\mathcal{H}_\infty$ norm of $u \to y$ and are shown in Figure 1 in red.

The computed value of γ using either Proposition III.1 or Proposition IV.1 is a guaranteed upper bound for the $\mathcal{H}_\infty$ norm of the true system for all noise levels. The bound provides a good approximation of γ_0 for low noise levels. For increasing noise levels, the bound γ becomes more conservative for both methods. Comparing the results from Proposition III.1 (unstructured data) with Proposition IV.1 (structured data), the bounds obtained from (9) are conservative with respect to those from (6) for higher noise levels, while the difference is small for low noise levels. By solving the unstructured data-based conditions in [12, Theorem 4], we find the same bounds as obtained per Proposition III.1, as expected from the duality of the results.

B. Example 2: Distributed $\mathcal{H}_\infty$ controller synthesis

Consider an interconnected system with $L = 25$ subsystems, each having one state ($n_i = 1$). The subsystems are interconnected according to a *cycle* graph $\mathcal{G}$ and the matrices A_i and A_{ij} are drawn uniformly on the interval $[0, 1]$ and $[0, 0.1]$, respectively, and $B_i = 1$. We consider $y_i = x_i$ for all subsystems and consider the performance output $z_i = x_i$, so that $C_i^z = I$ and $C_{ij}^z = D_i^z = 0$. For the data acquisition, the input entries are drawn from a normal distribution with zero mean and unit variance. The noise signals $w_i(k)$ are drawn uniformly from the set $\{w \mid |w| \leq \sigma\}$, where $\sigma = 0.05$ is the noise level. Hence, considering the sets $\mathcal{W}_i$ with $Q_w^i = -I$, $S_w^i = 0$ and $R_w^i = N\sigma^2 I$, we have that the noise sequences satisfy $W_-^i \in \mathcal{W}_i$, $i = 1, \ldots, L$.

The goal is to synthesize a distributed controller that yields an upperbound γ on the $\mathcal{H}_\infty$ norm of the channel $w \to z$, without using knowledge of A_i, A_{ij} and B_i. First, we verify what the smallest upperbound γ is, for which there

$$\Psi_i^\top (\star)^\top \left[\begin{array}{cc|cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_i R_\mathcal{D}^i & -\alpha_i (S_\mathcal{D}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_i S_\mathcal{D}^i & -\alpha_i Q_\mathcal{D}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & I \\ \hline \mathbf{1} \otimes I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 & I \\ C_i^z & C_{\mathcal{N}_i}^z & 0 & 0 \end{array}\right] \Psi_i < 0 \quad (12)$$

$$\Phi_i^\top (\star)^\top \left[\begin{array}{cc|cc|cc|cc} -\bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{Z}_i^{11} & \bar{Z}_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (\bar{Z}_i^{12})^\top & \bar{Z}_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\beta_i \bar{R}_\mathcal{D}^i & -\beta_i (\bar{S}_\mathcal{D}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_i \bar{S}_\mathcal{D}^i & -\beta_i \bar{Q}_\mathcal{D}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^{-2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & I \\ \hline \mathbf{1} \otimes I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 & I \\ C_i^z & C_{\mathcal{N}_i}^z & 0 & 0 \end{array}\right]_\perp \Phi_i > 0 \quad (13)$$

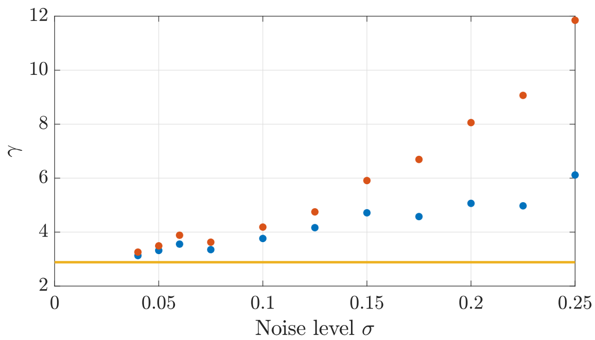

Fig. 1. Example 1: Upper bound on the $\mathcal{H}_\infty$ norm determined from noisy data with increasing noise levels σ via lumped (blue) and structured (red) data analysis. The $\mathcal{H}_\infty$ norm of the true system is shown in orange.

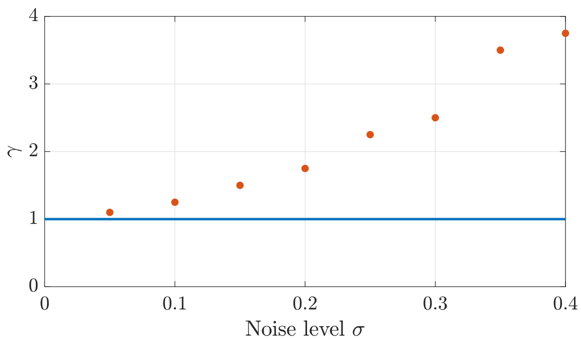

Fig. 2. Example 2: Achievable $\mathcal{H}_\infty$ norm with distributed control from noisy data with increasing noise levels σ (red) and achievable $\mathcal{H}_\infty$ norm with distributed control computed using the true system (blue).

exists a *model-based* distributed controller by the *nominal* LMIs in [16, Theorem 2]. This smallest upperbound of γ is 1.00 and serves as a benchmark: our data-based method for distributed control cannot perform better than the model-based distributed controller. We generate the data matrices (U_i^-, X_i) for $N = 50$ samples. For $\alpha_i := \alpha = 1$, we observe that the LMIs (12) and (13) are feasible for $\gamma = 1.10$. Hence, by Theorem V.1, there exists a distributed controller that achieves an $\mathcal{H}_\infty$ norm less than 1.10 in closed-loop with the true interconnected system.

Next, we increase the noise level, up to $\sigma = 0.4$. The resulting values of γ are shown in Figure 2 and obtained from the conditions in Theorem V.1 by varying α in a discrete interval. We observe that the conservatism increases for increasing noise levels. This can be explained by the increasing size of $\Sigma_\mathcal{D}^i$, leading to the existence of a more conservative distributed controller that achieves $\mathcal{H}_\infty$ performance for all interconnected systems consistent with the data.

VII. Concluding Remarks

We have considered the problem of analyzing the $\mathcal{H}_\infty$ norm of an interconnected system and finding a distributed controller that achieves $\mathcal{H}_\infty$ performance based on noisy data. First, we considered a dual parameterization of the set of systems consistent with the data and we presented a dual result for data-based dissipativity analysis, with respect to the results in [12]. A dual parameterizations of data-compatible subsystems allowed us to introduce an interconnected system with LFT representations of the subsystems. We have presented sufficient LMI conditions based on data that guarantee $\mathcal{H}_\infty$ performance or the existence of a distributed controller that achieves $\mathcal{H}_\infty$ performance.

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