

Local module identification in dynamic networks with correlated noise: the full input case

Paul M.J. Van den Hof, Karthik R. Ramaswamy, Arne G. Dankers and Giulio Bottegal

Abstract—The identification of local modules in dynamic networks with known topology has recently been addressed by formulating conditions for arriving at consistent estimates of the module dynamics, typically under the assumption of having disturbances that are uncorrelated over the different nodes. The conditions typically reflect the selection of a set of node signals that are taken as predictor inputs in a MISO identification setup. In this paper an extension is made to arrive at an identification setup for the situation that process noises on the different node signals can be correlated with each other. In this situation the local module may need to be embedded in a MIMO identification setup for arriving at a consistent estimate with maximum likelihood properties. This requires the proper treatment of confounding variables. The result is an algorithm that, based on the given network topology and disturbance correlation structure, selects an appropriate set of node signals as predictor inputs and outputs in a MISO or MIMO identification setup. As a first step in the analysis, we restrict attention to the (slightly conservative) situation where the selected output node signals are predicted based on all of their in-neighbor node signals in the network.

I. INTRODUCTION

In recent years increasing attention has been given to the development of new tools for the identification of large-scale interconnected systems, also known as dynamic networks. These networks are typically thought of as a set of measurable signals (the node signals) interconnected through linear dynamic systems (the modules), possibly driven by external excitations (the reference signals). Among the literature on this topic, we can distinguish three main categories of research. The first one focuses on identifying the topology of the dynamic network [1], [2], [3], [4]. The second category concerns identification of the full network dynamics [5], [6], [7], [8] while the third one deals with identification of a specific component (module) of the network, assuming that the network topology is known (the so called local module identification, see [9], [10], [11], [12], [13]).

In this paper we will further expand the work on the local module identification problem. In [9], the classical *direct-method* [14] for closed-loop identification has been generalized to a dynamic network framework using a MISO identification setup. Consistent estimates of the target module

can be obtained when the network topology is known and all the node signals in the MISO identification setup are measured. The work has been extended in [15] towards the situation where some node signals might be non-measurable, leading to an additional predictor input selection problem. A similar setup has also been studied in [10], where an approach has been presented based on empirical Bayesian methods to reduce the variance of the target module estimates. In [13] and [11], dynamic networks having node measurements corrupted by sensor noise have been studied, and informative experiments for consistent local module estimates have been addressed in [12].

A standing assumption in the aforementioned works [9], [10], [12], [15] is that the process noises entering the nodes of the dynamic network are uncorrelated with each other. This assumption facilitates the analysis and the development of methods for local module identification, reaching *consistent* module estimates using the direct method. However, when process noises are correlated over the nodes, the consistency results for the considered MISO direct method collapse. In this situation it seems necessary to consider also the *noise topology or disturbance correlation structure*, when selecting an appropriate identification setup. Even though the two-stage methods in [13] and [11] can handle the situation of correlated noise and deliver consistent estimates, the obtained estimates will not have minimum variance.

In this paper we precisely consider the situation of having dynamic networks with disturbance signals on different nodes that possibly are correlated, while our target moves from consistency only, to also minimum variance (or Maximum Likelihood (ML)) properties of the obtained estimates. While one could use techniques for full network identification (e.g., [5]), our aim is to develop a method that uses only local information. In this way, we avoid (i) the need to collect node measurements that are “far away” from the target module, and (ii) the need to identify unnecessary modules that would come with the price of higher variance in the estimates.

Using the reasoning first introduced [16], we build a constructive procedure that, choosing a limited number of predictor inputs and predicted outputs, builds an identification setup that guarantees maximum likelihood (ML) properties (and thus asymptotic minimum variance) when applying a direct prediction error identification method. In this situation we have to deal with so-called *confounding variables* (see e.g. [16], [17]), that is, unmeasured variables that directly or indirectly influence both the predicted output and the predictor inputs, and lead to lack of consistency.

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A direct influence, caused by correlated process noise, can be treated by adding predicted outputs to our identification setting, while an indirect influence, caused by unmeasured nodes, can be resolved by adding predictor inputs. In this paper, we restrict our attention to the situation where all the nodes that are in-neighbors of predicted outputs are measured, which we refer to as the *full input case*.

This paper is organized as follows. In section II, the dynamic network setup is defined. Section III provides a summary of available results from the existing literature of local module identification related to the context of this paper. Next, important concepts and notations used in this paper are defined in Section IV. Section V provides an algorithm for selecting the predictor inputs and predicted outputs while the MIMO identification setup and predictor model are provided in the next section. Section VII presents the main results of this paper followed by two illustrative examples of the introduced method in the subsequent section. Conclusions are discussed in section IX.

II. NETWORK SETUP

Following the basic setup of [9], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j , $j = 1, \dots, L$, and K *external variables* r_k , $k = 1, \dots, K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^L G_{jl}(q)w_l(t) + r_j(t) + v_j(t) \quad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl} is a proper rational transfer function matrix, and the single transfer functions G_{jl} are referred to as *modules*.
- r_j are *external variables* that can directly be manipulated by the user and that may or may not be present; if r_j is not present it is replaced by $r_j = 0$.
- v_j is *process noise*, where the vector process $v = [v_1 \dots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density, such that there exists a white noise process $e := [e_1 \dots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase.

In line with the situations considered in [7], we will assume that either all modules in G are strictly proper, or that Λ is restricted to be diagonal.

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1 L} \\ G_{L1} & \cdots & G_{L L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + r + He. \quad (2)$$

The identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of measured variables w , and possibly r .

III. AVAILABLE RESULTS

We address the so-called local identification problem of estimating one single module on the basis of measured node variable signals and possibly external excitation signals. In the current approaches to this problem, attention has been given to the selection of predictor input variables, when the target is to identify module G_{ji}^0 consistently.

Let us define \mathcal{N}_j as the set of node indices k such that $G_{jk} \neq 0$, i.e. the node signals in \mathcal{N}_j are the in-neighbors of the node signal w_j . Let \mathcal{D}_j denote the set of indices of the internal variables that are chosen as predictor inputs. Let \mathcal{Z}_j denote the set of indices not in $\{j\} \cup \mathcal{D}_j$, i.e. $\mathcal{Z}_j = \{1, \dots, L\} \setminus \{\{j\} \cup \mathcal{D}_j\}$. Let $w_{\mathcal{D}}$ denote the vector $[w_{k_1} \dots w_{k_n}]^T$, where $\{k_1, \dots, k_n\} = \mathcal{D}_j$. Let $r_{\mathcal{D}}$ denote the vector $[r_{k_1} \dots r_{k_n}]^T$, where $\{k_1, \dots, k_n\} = \mathcal{D}_j$, and where the ℓ th entry is zero if r_{ℓ} is not present in the network (i.e. $\ell \notin \mathcal{R}$). The vectors $w_{\mathcal{Z}}$, $v_{\mathcal{D}}$, $v_{\mathcal{Z}}$ and $r_{\mathcal{Z}}$ are defined analogously. The ordering of the elements of $w_{\mathcal{D}}$, $v_{\mathcal{D}}$, and $r_{\mathcal{D}}$ is not important, as long as it is the same for all vectors. The transfer function matrix between $w_{\mathcal{D}}$ and w_j is denoted $G_{j\mathcal{D}}^0$. The other transfer function matrices are defined analogously. By this notation, the network equations (2) is rewritten as:

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G_{j\mathcal{D}}^0 & G_{j\mathcal{Z}}^0 \\ G_{\mathcal{D}j}^0 & G_{\mathcal{D}\mathcal{D}}^0 & G_{\mathcal{D}\mathcal{Z}}^0 \\ G_{\mathcal{Z}j}^0 & G_{\mathcal{Z}\mathcal{D}}^0 & G_{\mathcal{Z}\mathcal{Z}}^0 \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \\ r_{\mathcal{Z}} \end{bmatrix}, \quad (3)$$

where $G_{\mathcal{D}\mathcal{D}}^0$ and $G_{\mathcal{Z}\mathcal{Z}}^0$ have zeros on the diagonal.

Identification of module G_{ji}^0 can now be done by selecting \mathcal{D}_j such that $i \in \mathcal{D}_j$, and subsequently estimating a multiple-input single output model including all modules G_{jk}^0 , $k \in \mathcal{D}_j$. This can be done by considering the prediction error

$$\begin{aligned} \varepsilon_j(t, \theta) &= w_j(t) - \hat{w}_j(t|t-1, \theta) \\ &= H_j(\theta)^{-1} \left(w_j - \sum_{k \in \mathcal{D}_j} G_{jk}(\theta)w_k - r_j \right) \end{aligned} \quad (4)$$

where arguments q and t have been dropped for notational clarity. The parameterized transfer functions $G_{jk}(\theta)$, $k \in \mathcal{D}_j$ and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors: $V_j(\theta) = \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_j^2(t, \theta)$, where N is the length of the data set. We refer to this identification method as the *direct method*, [9]. Let $\hat{\theta}_N$ denote the minimizing argument of $V_j(\theta)$.

The following results are available from previous work:

- When \mathcal{D}_j is chosen equal to \mathcal{N}_j and noise v_j is uncorrelated to all v_k , $k \in \mathcal{D}_j$, then G_{ji}^0 can be consistently estimated in a MISO setup, provided that there is enough excitation in the predictor input signals, see [9].
- When \mathcal{D}_j is a subset of \mathcal{N}_j , confounding variables¹ can occur in the estimation problem, and these have to be taken into account in the choice of \mathcal{D}_j in order to arrive at consistent estimates of G_{ji}^0 , see [15]. This situation has been analyzed for uncorrelated disturbances only, i.e. Φ_v being diagonal.

¹A confounding variable is an unmeasured variable that is correlated to both the input and output of an estimation problem. [18]

- In [17] relaxed conditions for the previous situation have been formulated, while still staying in the context of MISO identification with Φ_v being diagonal. This is particularly done by choosing additional predictor input signals that are not in \mathcal{N}_j , i.e. that are no in-neighbors of the output w_j of the target module.
- Irrespective of noise correlations, an indirect/two-stage identification method can be used to arrive at consistent estimates of G_{ji}^0 , if particular conditions on \mathcal{D}_j are satisfied, [9], [15]. However the drawback of indirect methods is that they do not allow for a maximum likelihood analysis, i.e. they will not lead to minimum variance results.

The step that we would like to make in this paper, is to go beyond consistency properties, and to formulate an identification setup that leads to Maximum Likelihood properties, and thus also minimum variance properties, of the estimated module, for the situation that the disturbance signals can be correlated, i.e. Φ_v not necessarily being diagonal. This requires a more careful treatment and modelling of the noise that is acting on the different node signals. In [16] a two-node example network has been studied, which has led to the following two suggestions:

- confounding variables can be dealt with by modelling correlated disturbances on the node signals, and
- this can be done by moving from a MISO identification setup to a MIMO setup.

These suggestions are being worked out in the current paper, and, as a first step in this analysis, we will stay in the situation of “full input modeling”, meaning that for every node signal that is included as a predicted output we will include all in-neighbors in the network as predictor input. A relaxation of this condition is left for future work. We will first present an example to explain the mechanism.

Example 1: Consider the network sketched in Figure 1, and let module G_{21} be the target module for identification.

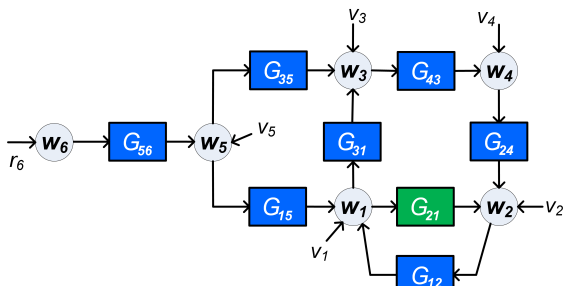


Fig. 1. Example network

If the node signals w_1 , w_2 and w_4 can be measured, then a two-input one-output model with inputs w_1, w_4 and output w_2 will (under the appropriate conditions) lead to a consistent estimate of G_{21} and G_{24} , provided that the disturbance signal v_2 is uncorrelated to the signals v_1 and v_4 . However if e.g. v_4 and v_2 are correlated, then consistency is lost for this approach. A solution is then to include w_4 in the set of predicted outputs, and by adding node signal w_3 as predictor input for w_4 . We then combine predicting w_2 on the basis of (w_1, w_4) with predicting w_4 on the basis of w_3 . The

correlation between v_2 and v_4 is then covered by modelling a 2×2 noise model of the joint process (v_2, v_4) .

In the next sections we will formalize the procedure as sketched in Example 1 for general networks.

IV. CONCEPTS AND NOTATION

Definition 1 (confounding variable): Given a dynamic network, with the interconnection structure defined by

$$w = Gw + He + r \quad (5)$$

with $cov(e) = I$, and consider the graph related to this network, with node signals w and e . Let $w_{\mathcal{D}}$ and $w_{\mathcal{Y}}$ be two subsets of node signals in w .

Then an unmeasured noise e_{ℓ} in e is a *confounding variable for the estimation problem* $w_{\mathcal{D}} \rightarrow w_{\mathcal{Y}}$, if there exist simultaneous paths from e_{ℓ} to node signals in $w_{\mathcal{D}}$ and $w_{\mathcal{Y}}$, while these paths are either direct or only run through nodes that are not in $w_{\mathcal{D}} \cup w_{\mathcal{Y}}$. \square

We will denote $w_{\mathcal{Y}}$ as the node signals in w that serve as predicted outputs, and $w_{\mathcal{D}}$ as the node signals in w that serve as predictor inputs. Next we decompose $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ in disjoint sets according to: $\mathcal{Y} = \mathcal{Q} \cup \{o\}$; $\mathcal{D} = \mathcal{Q} \cup \mathcal{A} \cup \mathcal{B}$ where $w_{\mathcal{Q}}$ are the node signals that are common in $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$; w_o is the output w_j of the target module; if $j \in \mathcal{Q}$ then $\{o\}$ is void; $\mathcal{A} \subset \mathcal{N}_j$ and $\mathcal{B} \not\subset \mathcal{N}_j$, to be specified later on. Additionally we denote $w_{\mathcal{Z}}$ as the node signals in w that are neither predicted output nor predictor input, i.e. $\mathcal{Z} = \mathcal{L} \setminus \{\mathcal{D} \cup \mathcal{Y}\}$, where $\mathcal{L} = \{1, 2, \dots, L\}$.

V. ALGORITHM FOR SIGNAL SELECTION: FULL INPUT CASE

In order to arrive at an appropriate identification setup we will take the following strategy:

- We start by constructing sets \mathcal{Q} and \mathcal{A} in such a way that all w -in-neighbors of $w_{\mathcal{Y}}$ are included in $w_{\mathcal{Q} \cup \mathcal{A}}$ and that all disturbance terms v_k , $k \in \mathcal{A}$ are uncorrelated to disturbance terms v_{ℓ} , $\ell \in \mathcal{Y}$.
- Then we choose $w_{\mathcal{B}}$ as a subset of nodes that are not in $w_{\mathcal{Y}}$ nor in $w_{\mathcal{A}}$. This set needs to be introduced to deal with confounding variables, and will be further specified in Section VII.
- Finally, we define the identification setup as the estimation problem $w_{\mathcal{D}} \rightarrow w_{\mathcal{Y}}$.

The conditions that need to be imposed on the selection of $w_{\mathcal{B}}$ in order to arrive at attractive properties of the estimation results, will be the main subject of analysis in this paper.

The following algorithm formalizes the procedure as indicated above.

Algorithm A

- 1) Select target module G_{ji}
- 2) Include j in the index set \mathcal{Y} of node variables that are to be predicted.
- 3) For every element x of \mathcal{Y} :
 - a) For every $k \in \mathcal{N}_x$:
 - include k in \mathcal{D} , and
 - if v_k is correlated with any w_{ℓ} , $\ell \in \mathcal{Y}$, then include k in \mathcal{Y} ;

b) If \mathcal{Y} has changed, start step 3 from the beginning again.

- 4) Determine \mathcal{Q} as the intersection of \mathcal{Y} and \mathcal{D} ;
- 5) If $j \notin \mathcal{Q}$ then set $w_o = w_j$, else w_o is void;
- 6) Determine $\mathcal{A} = \mathcal{D} \setminus \mathcal{Q}$;
- 7) Make a selection \mathcal{B} of node signals that are not in \mathcal{Y} and not in \mathcal{A} .

When this algorithm finishes, then the set \mathcal{Y} contains the index set of to be predicted node variables, while for each predicted node variable x in this set, the set of predictor inputs is \mathcal{N}_x . In this procedure, input nodes with disturbances that are uncorrelated to output disturbances, will block the further growth of the number of signals in \mathcal{Q} and \mathcal{A} , while input nodes with correlated disturbances will require further extension of the sets.

VI. MIMO IDENTIFICATION SETUP

On the basis of the decomposition of node signals as defined in the previous section we are going to rewrite the system's equations (5) in the following structured form:

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{Q}\mathcal{Q}} & G_{\mathcal{Q}o} & G_{\mathcal{Q}\mathcal{B}} & G_{\mathcal{Q}\mathcal{A}} & G_{\mathcal{Q}\mathcal{Z}} \\ G_{o\mathcal{Q}} & G_{oo} & G_{o\mathcal{B}} & G_{o\mathcal{A}} & G_{o\mathcal{Z}} \\ G_{\mathcal{B}\mathcal{Q}} & G_{\mathcal{B}o} & G_{\mathcal{B}\mathcal{B}} & G_{\mathcal{B}\mathcal{A}} & G_{\mathcal{B}\mathcal{Z}} \\ G_{\mathcal{A}\mathcal{Q}} & G_{\mathcal{A}o} & G_{\mathcal{A}\mathcal{B}} & G_{\mathcal{A}\mathcal{A}} & G_{\mathcal{A}\mathcal{Z}} \\ G_{\mathcal{Z}\mathcal{Q}} & G_{\mathcal{Z}o} & G_{\mathcal{Z}\mathcal{B}} & G_{\mathcal{Z}\mathcal{A}} & G_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} H_{\mathcal{Q}\mathcal{Q}} & H_{\mathcal{Q}o} & H_{\mathcal{Q}\mathcal{B}} & H_{\mathcal{Q}\mathcal{A}} & H_{\mathcal{Q}\mathcal{Z}} \\ H_{o\mathcal{Q}} & H_{oo} & H_{o\mathcal{B}} & H_{o\mathcal{A}} & H_{o\mathcal{Z}} \\ H_{\mathcal{B}\mathcal{Q}} & H_{\mathcal{B}o} & H_{\mathcal{B}\mathcal{B}} & H_{\mathcal{B}\mathcal{A}} & H_{\mathcal{B}\mathcal{Z}} \\ H_{\mathcal{A}\mathcal{Q}} & H_{\mathcal{A}o} & H_{\mathcal{A}\mathcal{B}} & H_{\mathcal{A}\mathcal{A}} & H_{\mathcal{A}\mathcal{Z}} \\ H_{\mathcal{Z}\mathcal{Q}} & H_{\mathcal{Z}o} & H_{\mathcal{Z}\mathcal{B}} & H_{\mathcal{Z}\mathcal{A}} & H_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{B}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix} \quad (6)$$

where we make the notation agreement that the matrix H is not necessarily monic, and the scaling of the white noise process e is such that $\text{cov}(e) = I$. Without loss of generality, we can assume $r = 0$ for the sake of brevity.

If we follow Algorithm A for the signal selection then we satisfy the following assumption.

Assumption 1: All w -in-neighbours of $w_{\mathcal{Y}}$ are collected in $w_{\mathcal{Q} \cup \mathcal{A}}$, and all disturbance signals $v_{\mathcal{A}}$ are uncorrelated to $v_{\mathcal{Y}}$.

Proposition 1: Under the conditions of Assumption 1 it follows that in (6), (a) $H_{\mathcal{A}\mathcal{Q}} = H_{\mathcal{A}o} = H_{\mathcal{Q}\mathcal{A}} = H_{o\mathcal{A}} = 0$ (b) $G_{\mathcal{Q}\mathcal{Z}} = G_{o\mathcal{Z}} = G_{\mathcal{Q}\mathcal{B}} = G_{o\mathcal{B}} = 0$ (c) $G_{oo} = 0$ (d) If w_o is present then $G_{\mathcal{Q}o} = 0$. \square

Due to lack of space, the detailed proof for the above proposition is provided in [19].

Proposition 2: Under the conditions of Assumption 1, the system equations for the measured variables $w_{\mathcal{D}} \cup w_{\mathcal{Y}}$ can be written as

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{Q}\mathcal{Q}} & 0 & 0 & G_{\mathcal{Q}\mathcal{A}} \\ G_{o\mathcal{Q}} & 0 & 0 & G_{o\mathcal{A}} \\ \check{G}_{\mathcal{B}\mathcal{Q}} & \check{G}_{\mathcal{B}o} & \check{G}_{\mathcal{B}\mathcal{B}} & \check{G}_{\mathcal{B}\mathcal{A}} \\ \check{G}_{\mathcal{A}\mathcal{Q}} & \check{G}_{\mathcal{A}o} & \check{G}_{\mathcal{A}\mathcal{B}} & \check{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \check{v}, \quad (7)$$

$$\check{v} = \check{H} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{B}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} H_{\mathcal{Q}\mathcal{Q}} & H_{\mathcal{Q}o} & H_{\mathcal{Q}\mathcal{B}} & 0 & H_{\mathcal{Q}\mathcal{Z}} \\ H_{o\mathcal{Q}} & H_{oo} & H_{o\mathcal{B}} & 0 & H_{o\mathcal{Z}} \\ \check{H}_{\mathcal{B}\mathcal{Q}} & \check{H}_{\mathcal{B}o} & \check{H}_{\mathcal{B}\mathcal{B}} & \check{H}_{\mathcal{B}\mathcal{A}} & \check{H}_{\mathcal{B}\mathcal{Z}} \\ \check{H}_{\mathcal{A}\mathcal{Q}} & \check{H}_{\mathcal{A}o} & \check{H}_{\mathcal{A}\mathcal{B}} & \check{H}_{\mathcal{A}\mathcal{A}} & \check{H}_{\mathcal{A}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{B}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix}$$

with $\text{cov}(e) = I$, and where

$$\check{G}_{\mathcal{A}\star} = G_{\mathcal{A}\star} + G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}G_{\mathcal{Z}\star}, \quad (8)$$

$$\check{H}_{\mathcal{A}\mathcal{Q}} = G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}\mathcal{Q}}, \quad (9)$$

$$\check{H}_{\mathcal{A}o} = G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}o}, \quad (10)$$

$$\check{H}_{\mathcal{A}\mathcal{B}} = H_{\mathcal{A}\mathcal{B}} + G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}\mathcal{B}}, \quad (11)$$

$$\check{H}_{\mathcal{B}\mathcal{B}} = H_{\mathcal{A}\mathcal{A}} + G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}\mathcal{A}}, \quad (12)$$

$$\check{H}_{\mathcal{A}\mathcal{Z}} = H_{\mathcal{A}\mathcal{Z}} + G_{\mathcal{A}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}\mathcal{Z}}, \quad (13)$$

$$\check{G}_{\mathcal{B}\star} = G_{\mathcal{B}\star} + G_{\mathcal{B}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}G_{\mathcal{Z}\star}, \quad (14)$$

$$\check{H}_{\mathcal{B}\star} = H_{\mathcal{B}\star} + G_{\mathcal{B}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_{\mathcal{Z}\star}. \quad (15)$$

\square

The detailed proof for the above proposition is provided in [19]. In the sequel we are going to formulate conditions on the choice of node variables in $w_{\mathcal{B}}$, such that the systems equations for the output variables in $w_{\mathcal{Y}}$ can be written as

$$\underbrace{\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \end{bmatrix}}_{w_{\mathcal{Y}}} = \underbrace{\begin{bmatrix} \bar{G}_{\mathcal{Q}\mathcal{Q}}^0 & \bar{G}_{\mathcal{Q}\mathcal{B}}^0 & \bar{G}_{\mathcal{Q}\mathcal{A}}^0 \\ \bar{G}_{o\mathcal{Q}}^0 & \bar{G}_{o\mathcal{B}}^0 & \bar{G}_{o\mathcal{A}}^0 \end{bmatrix}}_{\bar{G}^0} \underbrace{\begin{bmatrix} w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix}}_{w_{\mathcal{D}}} + \underbrace{\begin{bmatrix} \bar{H}_{\mathcal{Q}\mathcal{Q}}^0 & \bar{H}_{\mathcal{Q}o}^0 \\ \bar{H}_{o\mathcal{Q}}^0 & \bar{H}_{oo}^0 \end{bmatrix}}_{\bar{H}^0} \underbrace{\begin{bmatrix} \xi_{\mathcal{Q}} \\ \xi_o \end{bmatrix}}_{\xi_{\mathcal{Y}}} \quad (16)$$

with $\xi_{\mathcal{Q}}$ and ξ_o white noise processes with dimensions conforming to $w_{\mathcal{Q}}$ and w_o , respectively, with $\text{cov}(\xi_{\mathcal{Y}}) = \bar{\Lambda}$ and with \bar{H}^0 being monic, stable and stably invertible. In the situation of a network system with the system's equations as in (16) we can set up a predictor model based on a parametrized model set determined by

$$\mathcal{M} := \{(\bar{G}(\theta), \bar{H}(\theta), \bar{\Lambda}(\theta)), \theta \in \Theta\},$$

while the actual data generating system is represented by $\mathcal{S} = (\bar{G}(\theta_o), \bar{H}(\theta_o), \bar{\Lambda}(\theta_o))$. The corresponding identification problem is defined by considering the one-step-ahead prediction of $w_{\mathcal{Y}}$, according to

$$\hat{w}_{\mathcal{Y}}(t|t-1) := \mathbb{E}\{w_{\mathcal{Y}}(t) \mid w_{\mathcal{Y}}^{t-1}, w_{\mathcal{D}}^t\}$$

where $w_{\mathcal{D}}^t$ denotes the past of $w_{\mathcal{D}}$, i.e. $\{w_{\mathcal{D}}(k), k \leq t\}$. The resulting prediction error becomes:

$$\varepsilon(t, \theta) := w_{\mathcal{Y}}(t) - \hat{w}_{\mathcal{Y}}(t|t-1; \theta) \quad (17)$$

$$= \bar{H}(q, \theta)^{-1} [w_{\mathcal{Y}}(t) - \bar{G}(q, \theta)w_{\mathcal{D}}(t)],$$

and the weighted least squares identification criterion

$$\hat{\theta}_N = \arg \min_{\theta} \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon^T(t, \theta) W \varepsilon(t, \theta), \quad (18)$$

with W any positive definite weighting matrix. This parameter estimate then leads to an estimated subnetwork $G_{\mathcal{Y}\mathcal{D}}(q, \hat{\theta}_N)$, with the estimated target module $G_{ji}(q, \hat{\theta}_N)$ as a component of this.

VII. MAIN RESULTS

First we will formulate conditions for the selection of the blocking node variables $w_{\mathcal{B}}$, that will allow to derive consistent identification results next.

Property 1: Let the node signals $w_{\mathcal{B}}$ be chosen to satisfy the following properties:

- 1) If there are no confounding variables for the estimation problem $w_A \rightarrow (w_Q, w_o)$, then \mathcal{B} is void implying that w_B is not present;
- 2) If there are confounding variables for the estimation problem $w_A \rightarrow (w_Q, w_o)$, then all of the following conditions are satisfied:
 - a. For any confounding variable for the estimation problem $w_A \rightarrow (w_Q, w_o)$, the paths from the confounding variable to a node signal w_A are blocked by a node signal in w_B , where the paths are either direct or unmeasured² ;
 - b. For all white noise sources e_k in e_B, e_A and e_Z there do not exist simultaneous paths from e_k to w_B and w_A , where the paths are either direct or unmeasured. Alternatively formulated: the nonmodelled disturbances on w_B and w_A are uncorrelated;
 - c. There are no direct or unmeasured paths from w_i to node variables in w_B ;
 - d. There are no direct or unmeasured paths from w_o to node variables in w_B ³. \square

Next we can formulate the main consistency result of this paper.

Theorem 1: Consider a (MIMO) network identification setup with predictor inputs w_D and predicted outputs w_Y , satisfying the conditions of Assumption 1 (full input case). Then a prediction error identification method according to (17)-(18), applied to a parametrized model set \mathcal{M} will provide a consistent estimate of the target module G_{ji}^0 , if

- 1) \mathcal{M} is chosen to satisfy $\mathcal{S} \in \mathcal{M}$;
- 2) The typical regularity conditions on the data that are required for convergence of the parameter estimate in prediction error identification, see [14], are satisfied;
- 3) The blocking node signals w_B are chosen to satisfy Property 1;
- 4) $\Phi_\kappa(\omega) > 0$ for a sufficiently high number of frequencies, where $\kappa(t) := [w_D^\top \quad \xi_Q^\top \quad w_o^\top]^\top$;
- 5) All the elements in $G_{QQ}, G_{QA}, G_{oQ}, G_{oA}$ are strictly proper (or) all existing path/loop from w_Q, w_o, w_B to w_Q and from w_Q, w_o, w_B to w_o have at least a delay. \square

The detailed proof for the above theorem is provided in [19]. There are typically two major conditions for arriving at consistency of the target module G_{ji} : one needs to be able to deal with confounding variables through the selection of an appropriate set of (blocking) node variables w_B that is included as predictor input, and there should be enough excitation present in the node signals, which actually reflects a type of identifiability property [7]. Note that this excitation condition may require that there are external excitation signals present at some locations, see also [20]. Note that since we are using a direct method for identification, the signals r are not directly used in the predictor model, although they serve the purpose of providing excitation in the network.

Remark 1: If we consider the excitation condition formulated in condition 4 of the Theorem, we see a slight

²An unmeasured path is a path that runs through nodes in w_Z only.

³Condition 2d drops, if w_j is in w_Q and w_o is void.

variation with respect to the classical condition for closed-loop systems, which typically would contain w_D and w_o in the vector signal. In the considered network situation where signals can be both input and output, the signal vector in condition 4 is extended with ξ_Q , i.e. the innovation signal related to the disturbances on node signals that are both input and output.

Since in the result of Theorem 1 we arrive at white innovation signals, the result can be extended to formulate Maximum Likelihood properties.

Theorem 2: Consider the situation of Theorem 1, and let the conditions for consistency be satisfied. Let ξ_Y be normally distributed, and let $\hat{\Lambda}(\theta)$ be parametrized independently from $\hat{G}(\theta)$ and $\hat{H}(\theta)$. Then, under zero initial conditions, the Maximum Likelihood estimate of θ^0 is

$$\hat{\theta}_N^{ML} = \arg \min_{\theta} \det \left(\frac{1}{N} \sum_{t=1}^N \varepsilon(t, \theta) \varepsilon^T(t, \theta) \right) \quad (19)$$

$$\Lambda(\hat{\theta}_N^{ML}) = \frac{1}{N} \sum_{t=1}^N \varepsilon(t, \hat{\theta}_N^{ML}) \varepsilon^T(t, \hat{\theta}_N^{ML}). \quad (20)$$

Proof: Can be shown by following a similar reasoning as in Theorem 1 of [5]. \square

VIII. SIMULATION EXAMPLES

In this section we will apply the developed local module identification methodology to two examples of dynamic networks. First we will consider the dynamic network in example 1 where v_2 and v_4 are mutually correlated while the other disturbance signals are uncorrelated with these and with each other. The target of identification is module G_{21} , and all node signals are available for measurements. Using the identification method developed in this paper, we first select the signals w_Q, w_o, w_A using the algorithm A. Since v_2 and v_4 are correlated we choose them both as outputs. Consequently, w_1, w_3 and w_4 are chosen as inputs, so that

$$\mathcal{Y} = \{2, 4\} \quad ; \quad \mathcal{D} = \{1, 4, 3\} \quad (21)$$

$$\mathcal{Q} = \mathcal{Y} \cap \mathcal{D} = \{4\} \quad ; \quad \mathcal{A} = \mathcal{D} \setminus \mathcal{Q} = \{1, 3\} \quad (22)$$

$$w_o = w_2. \quad (23)$$

For the selection of w_B , according to Property 1, we need to check the presence of confounding variables. Since all disturbance terms $v_k, k \in \mathcal{Z} \cup \mathcal{A}$ are uncorrelated to all disturbance terms $v_l, l \in \mathcal{Y}$, there are no confounding variables for the estimation problem $w_A \rightarrow (w_Q, w_o)$. Therefore w_B is void. Now we have the predictor inputs w_D and the predicted outputs w_Y for the MIMO identification setup that will satisfy the essential conditions of Theorems 1 and 2.

Example 2: Consider the network sketched in Figure 2, and let module G_{12} be the target module for identification. The disturbance correlation structure in the network is presented in Figure 2 with modules in red indicating the noise dynamics.

The direct method using a MISO predictor, as addressed in [9], does not provide a consistent estimate of G_{12} since the disturbance term v_1 is correlated with v_2 as well as v_3 and therefore we resort to the identification framework developed

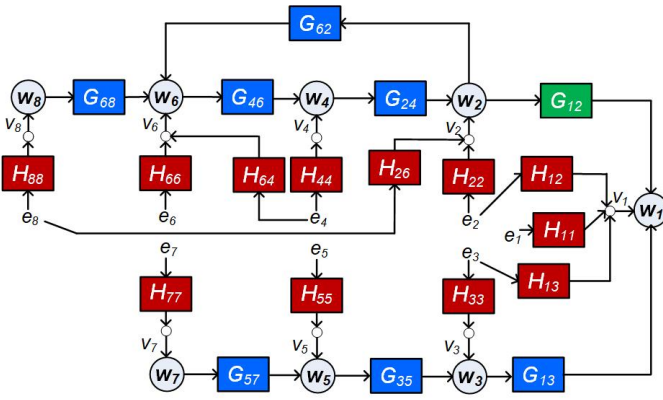


Fig. 2. Example network

in this paper. Similar to the previous example, the first step will be selection of $w_{\mathcal{Q}}, w_{\mathcal{O}}, w_{\mathcal{A}}$ using the algorithm A.

First we select w_1 as output and w_2 and w_3 as inputs. Since v_2 and v_3 are correlated with v_1 , both w_2 and w_3 need to be added as outputs too. Then w_4 and w_5 need to be added as inputs. As a result of the first six steps in algorithm A we get,

$$\mathcal{Y} = \{1, 2, 3\} \quad ; \quad \mathcal{D} = \{2, 3, 4, 5\} \quad (24)$$

$$\mathcal{Q} = \mathcal{Y} \cap \mathcal{D} = \{2, 3\} \quad ; \quad \mathcal{A} = \mathcal{D} \setminus \mathcal{Q} = \{4, 5\} \quad (25)$$

$$w_{\mathcal{O}} = w_1. \quad (26)$$

In the resulting situation e_8 acts as a confounding variable that affects both input w_4 and output w_2 . As per condition 2a of Property 1, the path from $e_8 \rightarrow w_4$ should be blocked by a node signal $w_{\mathcal{B}}$, which can be either w_8 or w_6 . In order to choose the node signals $w_{\mathcal{B}}$, we also need the conditions 2b, 2c and 2d in Property 1 to be satisfied. w_6 cannot be chosen in $w_{\mathcal{B}}$ since it does not satisfy conditions 2b and 2c in Property 1. The former condition is not satisfied due to the simultaneous path from e_4 in $e_{\mathcal{A}}$ to w_6 and w_4 and the latter condition is not satisfied due to the path from w_2 in $w_{\mathcal{I}} \rightarrow w_6$. When w_8 is chosen in $w_{\mathcal{B}}$, the conditions in Property 1 are satisfied and hence $\mathcal{B} = \{8\}$. Now we have the predictor inputs $w_{\mathcal{D}}$ and the predicted outputs $w_{\mathcal{Y}}$ for the MIMO identification setup that provide the consistent and maximum likelihood estimation results of G_{12} .

IX. CONCLUSIONS

A new local module identification approach has been presented to identify local modules in a dynamic network with given topology, addressing the situation that process noise on different nodes can be correlated with each other. For this case, it is shown that the problem can be solved by moving from a MISO to a MIMO identification setup. In this setup the target module is embedded in a MIMO problem with appropriately chosen inputs and outputs, that warrant the consistent estimation of the target module with maximum likelihood properties. A key part of the procedure is the handling of confounding variable, through the introduction of appropriately chosen additional predictor input node signals (blocking nodes). We have considered the "full input" case, implying that all in-neighbours of an output node are

included as input. A further relaxation of this condition is a future step that needs to be made. The presented approach has been illustrated by two examples.

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