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Bang-Bang Control in Reservoir Flooding

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SUMMARY

Various studies have shown that dynamic optimization of waterflooding using optimal control theory has a significant potential to increase Net Present Value (NPV). In these studies, gradient-based optimization methods are used, where the gradients are usually obtained with an adjoint formulation. However, the shape of the optimal injection and production settings is generally not known beforehand. The main contribution of this paper is to show that a whole variety of reservoir flooding problems can be formulated as optimal control problems that are linear in the control and that, if the only constraints are upper and lower bounds on the control, these problems will sometimes have bang-bang (on-off) optimal solutions. This is supported by a waterflooding example of a 3-dimensional reservoir in a fluvial depositional environment, modeled with 18.553 grid blocks. The valve settings of 8 injection and 4 production wells are optimized over the life of the reservoir, with the objective to maximize NPV. For various situations, the optimal solution is either bang-bang, or a bang-bang solution exists that is only slightly suboptimal. This has obvious practical implications, since bang-bang solutions can be implemented with simple on-off control valves.

1. Introduction

1.1 Reservoir flooding as optimal control problem

A whole variety of reservoir flooding problems can be formulated as optimal control problems, where the goal is to find a control $\mathbf{u}$ over a time interval $[0,T]$ that maximizes a certain performance measure $J(\mathbf{u})$. What distinguishes an optimal control problem from a static optimization problem is that $\mathbf{u}$ drives a dynamical system, often described by differential equations together with an initial condition, whose subsequent state trajectory determines $J(\mathbf{u})$.

In reservoir flooding, the controls are often combinations of injection rates, production rates, bottom hole pressures, concentrations and / or valve-settings, while the performance measure is often Net Present Value (NPV). The dynamical system is of course a reservoir. Over the past few years, dynamic optimization of reservoir flooding using optimal control theory has received significant attention. Various studies have shown that dynamic (time-varying) injection and production settings can yield a higher NPV than conventional reactive settings.

1.2 Literature review

Asheim (1987) and Asheim (1988) consider optimizing individual injection and production rates in water flooding. Ramirez and Fathi (1984), Fathi and Ramirez (1984, 1986, 1987) consider optimizing injection concentrations in chemical flooding. Mehos and Ramirez (1989) consider optimizing the production bottom hole pressure and the injection rates of water and CO_2 in CO_2 flooding. Liu et al. (1993) consider optimizing the rate and steam quality of an injector and the bottom hole pressure of a producer in steam flooding.

Even though the particular applications in these studies vary, they have much in common. The goal is always to maximize NPV, while the gradients of NPV with respect to the controls are computed using an adjoint model. Furthermore, the shapes of the optimal solutions turn out to be smooth (*i.e.* gradually varying; continuously differentiable), but it is not clear why.

Sudaryanto and Yortsos (2000) and Sudaryanto and Yortsos (2001) consider maximizing water breakthrough time in water flooding by optimizing two injection rates. However, in contrast to the previously mentioned literature, they state that the optimal solution is likely to be a *bang-bang control* - meaning that over the entire time interval, each component of $\mathbf{u}$ takes on either its minimum or maximum value. Consequently, they disregard the possibility of smooth optimal solutions and only consider bang-bang controls, parameterized in terms of switching times.

In light of this work, the subsequent studies (Brouwer 2004) and (Brouwer and Jansen 2004), which consider optimizing individual rates and valve settings in water flooding, are particularly interesting. They find that the optimal rates are smooth, but that the optimal valve settings are sometimes bang-bang - even though they do not parameterize the control in terms of switching times. No explanation is given as to the discriminating factor.

1.3 Main contribution and outline of paper

The main contribution of this paper is to investigate why and under what conditions reservoir flooding problems can be expected to have bang-bang optimal solutions.

A major practical advantage of bang-bang controls is of course that they can be implemented with simple on-off valves.

As a prerequisite for later sections, Section 2 shows that many reservoir flooding problems are in fact *linear* in the control (but nonlinear in the state), meaning that both the differential equations governing the dynamics of the reservoir model and the to-be-optimized performance measure depend linearly on the control. Then, why and under what conditions the optimal solutions to such problems can be expected to be bang-bang is clarified in Section 3. In Section 4, these results are illustrated by a water flooding example of a three dimensional oil-water reservoir containing 12 wells, modelled with 18.553 grid blocks.

2. Reservoir flooding problems linear in the control

2.1 Reservoir models linear in the control

The common method for numerical modelling of fluid flow through porous media is combining a mass-balance equation with Darcy's Law, which states that flow is proportional to a pressure gradient – (Peaceman 1977), (Aziz and Settari 1979). The resulting equations can, after some manipulation, be replaced by a single equation of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}_1\left(\mathbf{x}(t),t\right) + \mathbf{f}_2\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t) \quad , \quad \mathbf{x}\left(0\right) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x}(t)$ is the state vector containing the pressures and saturations in all grid blocks and $\mathbf{x}_0$ is the initial condition - (Aziz and Settari 1979), (Brouwer 2004). Here, $\mathbf{u}(t)$ is the control vector and contains the variables that we can directly manipulate, such as flow rates, bottom hole pressures or valve settings.

2.2 Performance measures linear in the control

A common economical performance measure is simple Net Present Value, defined as the total oil revenues minus the total injection and production costs over a time interval $[0,T]$, in combination with a discount factor. Another commonly used performance measure is the sweep efficiency. Both performance measures can be written in the form

$$J(\mathbf{u}) = \int_0^T \left(l_1\left(\mathbf{x}(t),t\right) + \mathbf{l}_2^T\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t)\right)\,dt\ , \tag{2}$$

where $\mathbf{u}$ enters the integrand of (2) linearly.

2.3 Constraints

In most applications there will be constraints on both the states and controls – (Sarma et al. 2006). For the moment, however, we only consider upper and lower bounds on the m individual components of the control.

In short, many reservoir flooding problems are in fact optimal control problems of the following form.

Problem 1:

$$\begin{aligned}
&\text{maximize} \quad J(\mathbf{u}) = \int_0^T \left(l_1\left(\mathbf{x}(t),t\right) + \mathbf{l}_2^T\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t)\right)\,dt \\
&\text{subject to} \quad \dot{\mathbf{x}}(t) = \mathbf{f}_1\left(\mathbf{x}(t),t\right) + \mathbf{f}_2\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t) \quad , \quad \mathbf{x}\left(0\right) = \mathbf{x}_0, \\
&\qquad\qquad \mathbf{u}(t) \in U := \left\{ \mathbf{z} \in \mathbb{R}^m : \mathbf{u}_{\min} \le \mathbf{z} \le \mathbf{u}_{\max} \right\} \quad \forall \quad t \in \left[0,T\right] .
\end{aligned}$$

3. Theory of bang-bang control

In this section we show that the optimal solution to ***Problem 1*** can be expected to be a bang-bang control by deriving first order necessary conditions for optimality. The basic reasoning behind deriving these conditions is that, given a candidate optimal control **u**, the first order variation of the performance measure should be nonpositive for 'small' variations of **u**. These derivations are given in virtually all textbooks on optimal control. See for example (Bryson and Ho 1975), (Luenberger 1981) and (Stengel 1986). A common approach is to define an auxiliary function, the so-called *Hamiltonian*, as follows

$$H\left(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right):=l_1\left(\mathbf{x}(t),t\right)+\mathbf{l}_2^T\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t)+\boldsymbol{\lambda}^T(t)\cdot\left(\mathbf{f}_1\left(\mathbf{x}(t),t\right)+\mathbf{f}_2\left(\mathbf{x}(t),t\right)\cdot\mathbf{u}(t)\right). \tag{3}$$

The vector $\boldsymbol{\lambda}(t)$ is often referred to as the adjoint vector. We consider arbitrary controls **v** that are allowable (meaning $\mathbf{v}(t) \in U\ \forall\ t \in [0,\mathrm{T}]$) and close to **u**, in the sense that the 'distance'

$$\delta:=\sum_{i=1}^{m}\int_0^T |\,\mathbf{v}_i(t)-\mathbf{u}_i(t)\,|\,dt \tag{4}$$

is small. It can be shown that if $\boldsymbol{\lambda}$ satisfies the *adjoint* system equation

$$\dot{\boldsymbol{\lambda}}(t)=-\frac{\partial H}{\partial \mathbf{x}}^T\left(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right)\ ,\ \ \boldsymbol{\lambda}\left(T\right)=0, \tag{5}$$

the effect on the performance measure is

$$J(\mathbf{v})-J(\mathbf{u})=\int_0^{\mathrm{T}}\left(H\left(\mathbf{x}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t),t\right)-H\left(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right)\right)\,dt+o(\delta) \tag{6}$$

where $o(\delta)$ denotes terms of smaller order than δ. A necessary condition for **u** to be optimal is therefore

$$H\left(\mathbf{x}(t),\mathbf{z},\boldsymbol{\lambda}(t),t\ \right)\ \le\ H\left(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right)\ \ \forall\ \mathbf{z}\in U,\ \forall\ t\in\left[0,T\right]. \tag{7}$$

This result is referred to as *Pontryagin's Maximum Principle* and it is one of the most important results in optimal control theory - see (Pontryagin et al. 1962). An important observation is that, for ***Problem 1***, these necessary conditions have a particular structure. With

$$\boldsymbol{\beta}^T\left(t\right):=\mathbf{l}_2^T\left(\mathbf{x}(t),t\right)+\boldsymbol{\lambda}^T(t)\cdot\mathbf{f}_2\left(\mathbf{x}(t),t\right) \tag{8}$$

the necessary condition (7) becomes

$$\boldsymbol{\beta}^T\left(t\right)\cdot\mathbf{z}\ \ \le\ \ \boldsymbol{\beta}^T\left(t\right)\cdot\mathbf{u}(t)\ \ \forall\ \mathbf{z}\in U,\ \forall\ t\in\left[0,T\right]. \tag{9}$$

This, in turn, necessarily leads to the following form of the components of **u**

$$\begin{aligned}\beta_i\left(t\right)<0\ &\Rightarrow\ u_i(t)=u_{\min,i}\\ \beta_i\left(t\right)>0\ &\Rightarrow\ u_i(t)=u_{\max,i}\end{aligned} \tag{10}$$

for $i = 1,\dots,m$. The function $\boldsymbol{\beta}$ is often understandably referred to as the *switching function* and its zeros the *switching times*. If the switching function contains only isolated zeros, the problem is said to be *regular*. On the other hand, if any component β_i is zero along an open time interval, the problem is said to be *singular*, and the associated trajectory segment is called a *singular arc*. The difficulty lies in the fact that along a singular arc, (7) no longer provides information on how to choose **u.**

*In short, any locally optimal solution to a reservoir flooding problem that can be written as **Problem 1** is necessarily a bang-bang control, possibly in combination with singular arcs.*

This is not surprising, as it is in line with the intuitive notion that in a local optimal solution of a static optimization problem, either the derivative vanishes or the to-be-optimized parameters are on the boundary of the feasible set. However, the following *is* surprising:

*Pure bang-bang controls (ones without singular arcs) are widely encountered as optimal strategy for optimal control problems that can be written as **Problem 1**.*

Some examples from other application fields include:
- minimum-time problems for linear systems – (Bellman 1956),
- minimum-time problems for bilinear systems – (Mohler 1973),
- optimal control of batch reactors – (Blakemore and Aris 1962),
- optimal thermal control – (Belghith et al. 1986),
- optimal drug administration in cancer chemotherapy – (Ledzewicz and Schattler 2002).

According to the previously mentioned studies by Sudaryanto and Yortsos, Brouwer and Jansen, and the example treated in Section 4, it now seems that pure bang-bang controls are also sometimes (but not always) encountered as optimal strategy for reservoir flooding problems.

Remark: It is now clear why we only consider upper and lower bounds on the individual components of the control **u**: the step from (9) to (10) in the derivation does not apply to problems involving more general (in)equality constraints on the control. In fact, this is *precisely* the reason why the optimal rates in (Brouwer 2004) and (Brouwer and Jansen 2004) are smooth, while the optimal valve settings are sometimes bang-bang: additional *equality* constraints are placed on the rates in order to balance total injection with total production, but not on the valve settings.

4. Example

4.1 Description of reservoir flooding problem

We consider a water flooding example of a 3-dimensional oil-water reservoir in a fluvial depositional environment. It is modeled with 18.553 grid blocks of dimension 20m x 20m x 20m, and there are 7 vertical layers. Figure 1 depicts the absolute horizontal permeability field, together with the location of 12 vertical wells.

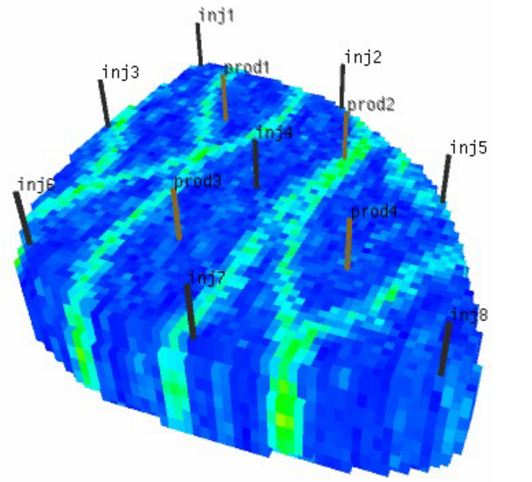

Figure 1: *Absolute horizontal permeability field, with 8 injectors and 4 producers.*

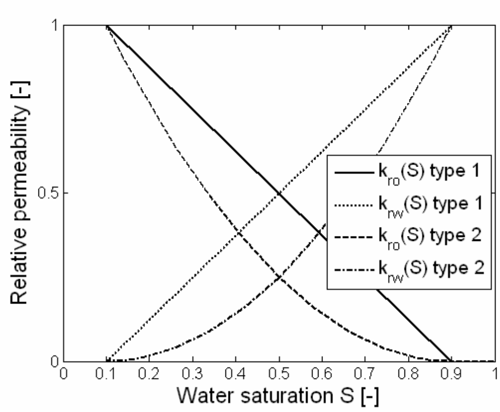

Figure 2: *Relative permeability curves.*

Symbol	Property	Value	Unit
φ	porosity	0.20	[-]
ρ_o(400 bar)	oil density	800 and 1000	[kg / m^3]
ρ_w(400 bar)	water density	1000	[kg / m^3]
c_o	oil compressibility	10^{-5}	[1 / bar]
c_w	water compressibility	10^{-5}	[1 / bar]
μ_o	oil viscosity	10^{-3}	[Pa s]
μ_w	water viscosity	10^{-3}	[Pa s]
p_{cow}	capillary pressure	0	[bar]
p_{bh}	bottom hole pressure	390-415	[bar]
p_{init}	initial pressure at top	400	[bar]
S_{init}	initial water saturation	0.10	[-]

Table 1: *Values of most of the geological and fluid properties used in the example.*

Most of the geological and fluid properties are given in Table 1. The reservoir geometry and values for the absolute permeability field can be obtained by contacting the corresponding author. Two different relative permeability curves are used, and are depicted in Figure 2.

The well specifications are as follows:

- Each well is available as of time $t=0$.
- Each well is vertical, and is perforated in all 7 layers of the reservoir.
- Each well operates at constant bottom-hole pressure. For the 8 injectors, the bottom hole pressure is set to 415 bar at the lowest perforation. For the 4 producers, the bottom hole pressure is set to 390 bar at the highest perforation. The pressures in the other perforations are computed assuming hydrostatic equilibrium in the wellbore.
- Each well is equipped with a single valve, whose setting can vary between 10^{-6} (a lower bound 0 leads to numerical problems) and 1: $U = \{ \mathbf{z} \in \mathbb{R}^{12} : 10^{-6} \leq z_i \leq 1, i = 1,\ldots,12 \}$.
- The valve setting of a well applies to all 7 perforations.
- The well indices are computed using a Peaceman model with a wellbore radius of 0.1m and zero skin factor.

The initial pressure in the reservoir is computed assuming hydrostatic equilibrium, with the top of the reservoir at a depth of 4000m and at a pressure of 400 bar. Note that due to the constant bottom hole pressures in the wells, the pressure in the reservoir always stays between 390 bar and 415 bar. The initial water saturation is 0.1 throughout the reservoir.

The performance measure is a simple NPV, defined as the undiscounted oil revenues minus the cost of water injection and production, using the values in Table 2. The goal is to maximize NPV by varying the valve settings of the 8 injectors and 4 producers over the interval $[0,T]$. Three different terminal times are considered: 1.5, 3.0, and 4.5 years. These reservoir flooding problems can be written as ***Problem 1***.

Property	Value	Unit
oil revenue	20	[$ / bbl]
water production cost	5	[$ / bbl]
water injection cost	0 and 1	[$ / bbl]
terminal time T	1.5, 3.0 and 4.5	[years]

Table 2: *Values for computing NPV.*

The previously described reservoir model is implemented in a proprietary reservoir simulator that also comprises the required adjoint model to compute switching functions (*i.e.* gradients). The differential equations described in Section 2 are approximated using a fully-implicit scheme, with a maximum time-step size of 1/50 year (≅1 week).

4.2 Results

The base case comparison is a conventional reactive water flooding strategy. Here, all valve settings are initially one, but a producer is shut in (and stays shut in) when it is no longer profitable to produce from it. With oil revenues at 20 [$ / bbl] and water production costs at 5 [$ / bbl], this profitability threshold corresponds to a water cut of 80%.

In order to find the optimal valve settings, a steepest descent method is used, with **u** divided into intervals of 1/50 year and initial guess 1. The effects of terminal time, oil density, relative permeability, and water injection cost on the shape of the optimal solution were investigated. The results are summarized in Table 3.

Case	T [years]	Type of relperm	Inj. cost [$ / bbl]	ρ_o [kg / m^3]	Type of control	Shape of control	NPV [billion $]	Increase [%]
1	1.5	2	0	800	reactive optimal	 bang-bang	1.350 1.365	 0.8
2	3.0	2	0	800	reactive optimal suboptimal	 singular bang-bang	1.871 1.934 1.933	 3.4 3.3
3	3.0	2	0	1000	reactive optimal suboptimal	 singular bang-bang	1.832 1.893 1.892	 3.3 3.3
4	3.0	2	1	800	reactive optimal suboptimal	 singular bang-bang	1.736 1.822 1.821	 5.0 4.9
5	3.0	1	0	800	reactive optimal suboptimal	 singular bang-bang	1.988 2.070 2.070	 4.1 4.1
6	4.5	1	1	1000	reactive optimal suboptimal	 singular bang-bang	1.900 2.140 2.119	 12.6 11.5

Table 3: *NPV and shape of reactive, optimal, and suboptimal controls.*

For some situations, the scheme indeed converges to a bang-bang control. We stress that the procedure *could* converge to a smooth solution, but does not because the optimal solution is apparently a bang-bang control. The optimal control for Case 1, denoted by $\mathbf{u}^*$, is depicted in Figure 3. The first order necessary optimality conditions are satisfied for $\mathbf{u}^*$, as can be seen by inspection of the sign of the corresponding switching function $\boldsymbol{\beta}^*$. At first sight, the strategy for the producers seems to be a reactive one. In fact this is not true: the second producer valve is set to zero at a water cut of 60% - far below the profitability threshold of 80%.

In other situations, the scheme does *not* converge to a bang-bang control, but to one with brief singular arcs for certain injectors. An example of such a situation is depicted in Figures 4 and 5. However, for each of these situations it was possible to find a *suboptimal bang-bang approximation* of the optimal control, with only a small loss in NPV. Obviously, this loss must be traded-off against the advantage of being able to implement the solution with simple on-off control valves.

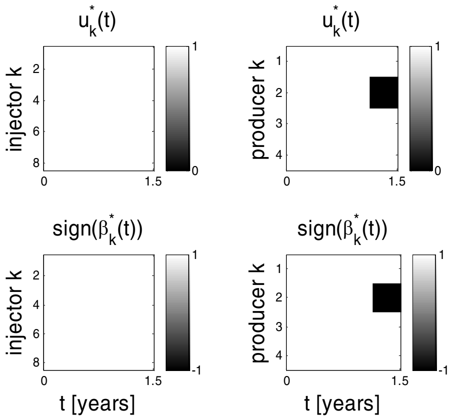

Figure 3: *Optimal valve settings* $\mathbf{u}^*$ *and sign of the switching function* $\boldsymbol{\beta}^*$ *for Case 1.*

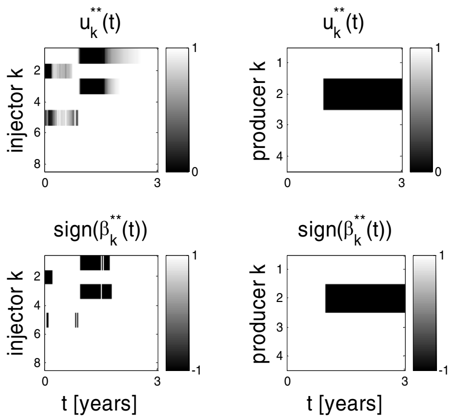

Figure 4: *Optimal valve settings* $\mathbf{u}^{**}$ *and sign of the switching function* $\boldsymbol{\beta}^{**}$ *for Case 2.*

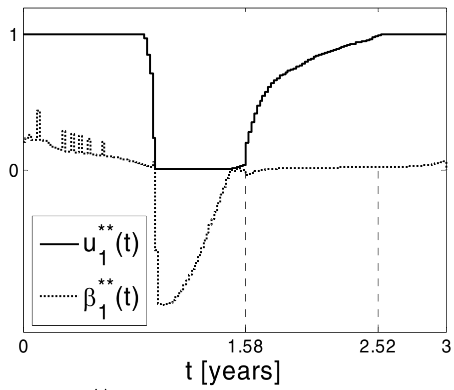

Figure 5: u_1^{**} *and (scaled) switching function* β_1^{**} *of injector 1 from* Figure 4; *there is a singular arc along* [1.58, 2.52].

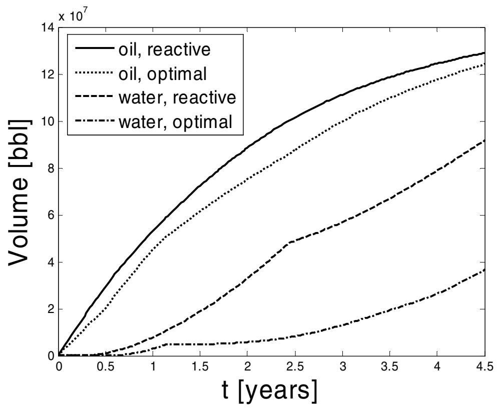

Figure 6: *Cumulative oil and water production of the reactive and optimal control for Case 6.*

The cumulative oil and water production of both strategies Case 6 are shown in Figure 6. Since there is no discount factor involved, the increase in NPV is solely due to a decrease in cumulative water production.

Remarks:

- Oil density does not seem to have a significant effect on the shape of the optimal solution. However, we do expect singular arcs to play a significant role in coning problems, where it is common to operate wells below the highest possible rate.
- Water injection costs do not seem to have a significant effect on the shape of the optimal solution.
- Later terminal times generally lead to more and longer singular arcs. It would be interesting to see if this still holds for problems with significant discounting in NPV.
- Problems with type 1 relative permeabilities (see Figure 2) generally have more and longer singular arcs than those with type 2 relative permeabilities.
- For Case 1, *several* (very similar) solutions were found that all satisfy the necessary conditions for optimality. The one shown in Figure 3 is the one with the highest NPV.
- There seems to be more scope for optimization in problems with later terminal times, type 1 relative permeabilities and, in particular, higher water injection costs.

A more detailed discussion is given in (Zandvliet et al. 2006).

5. Conclusions

Many reservoir flooding problems can be written as optimal control problems that are linear in the control. If the only constraints are upper and lower bounds on the control, due to their particular structure, these problems will sometimes have bang-bang optimal solutions. This is supported by a water flooding example, where for various situations the optimal solution is either bang-bang, or a bang-bang solution exists that is only slightly suboptimal. This has obvious practical implications, since bang-bang solutions can be implemented with simple on-off control valves.

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