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Excitation allocation for generic identifiability of a single module in dynamic networks: A graphic approach ⋆

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Abstract: For identifiability of a single module in a dynamic network, excitation signals need to be allocated at particular nodes in the network. Current techniques provide analysis tools for verifying identifiability in a given situation, but hardly address the synthesis question: where to allocate the excitation signals to achieve generic identifiability. Starting from the graph topology of the considered network model set, a new analytic result for generic identifiability of a single module is derived based on the concept of *disconnecting sets*. For the situation that all node signals are measured, the vertices in a particular disconnecting set provide the potential locations to allocate the excitation signals. Synthesis approaches are then developed to allocate excitation signals to guarantee generic identifiability.

Keywords: System identification, identifiability, dynamic networks, graph theory

1. INTRODUCTION

Due to the increasing complexity of current technological systems, the study of large-scale interconnected dynamic systems receives considerable attention. As a modeling framework for dynamic networks, we consider the network of transfer functions introduced in (Gonçalves and Warnick, 2008; Van den Hof et al., 2013), where vertices represent the internal signals, and directed edges denote transfer functions which are called modules. Identification problems in this setup involve multiple aspects, including estimation of one local module (Van den Hof et al., 2013; Dankers et al., 2015; Materassi and Salapaka, 2019), estimation of the topology (Materassi and Innocenti, 2010; Shi et al., 2019), estimation of the full network model (Weerts et al., 2018b) and identifiability of the models (Weerts et al., 2018a; Hendrickx et al., 2019; Cheng et al., 2019a).

The analysis of identifiability of dynamic networks aims for conditions under which network models in a model set can be distinguished based on data. This analysis provides guidelines for users to choose an appropriate model set such that the identification problem has a unique solution. Two notions of identifiability have been addressed in the literature, including *global identifiability* and *generic identfiability*. On the one hand, global identifiability requires models to be distinguishable from all other models in the model set (Weerts et al., 2018a; van Waarde et al., 2018). On the other hand, generic identifiability requires that almost all models can be distinguished from the other models in the set (Bazanella et al., 2017; Hendrickx et al., 2019; Weerts et al., 2018c). The advantage of considering generic identifiability is that the theoretical results can typically be formulated into graphical conditions on the network structure. This formulation can lead to efficient graphical approaches to analyze generic identifiability.

In addition, two different practical settings are considered for the study of generic identifiability. In one setting, all node signals can be measured and a subset of nodes is excited (Weerts et al., 2018c; Cheng et al., 2019a), while also allowing excitation through noise signals; while in the other setting, a subset of node signals is measured and all nodes can be excited while excitation is provided through external excitation only (Bazanella et al., 2017; van Waarde et al., 2018; Hendrickx et al., 2019). A recent contribution (Bazanella et al., 2019) extends the later work to partial measurement and partial excitation.

For generic identifiability when all node signals are measured, analysis results for generic identifiability of a single module can be found in (Weerts et al., 2018c), where both algebraic and graphical conditions have been developed. However, the results do not provide a structured method to allocate excitation signals for generic identifiability. In (Cheng et al., 2019b), an approach to allocate signals is developed for generic identifiability of a full network based on the pseudotree covering; however, this approach is hardly applicable to a single module. In this work, given a network model set and assuming users have the freedom to assign excitation signals, the research question is how to allocate the excitation signals such that one single module of a network becomes generically identifiable.

This research question is addressed by deriving a new analytic result for generic identifiability of a single module using the concept of disconnecting sets. Given the output of a target module, it is further found that the vertices in the disconnecting set from the input of the module to

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the other inputs of the output provide the potential locations for allocating the excitation signals. Then synthesis approaches are developed to allocate excitation signals at vertices in the disconnecting such that generic identifiability of a single module is guaranteed.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Dynamic networks and graphical representation

A dynamic network encompasses the relationship among scalar-valued *internal signals* w_j, deterministic excitation signals r_k which can be manipulated by the user, and unmeasured disturbances e_l, where $j \in \{1, \cdots, L\}$, $k \in \{1, \cdots, K\}$, $l \in \{1, \cdots, p\}$ and $p \leqslant L$. Following (Van den Hof et al., 2013), the interconnection among the signals is modeled using linear time-invariant systems, and for any j, the equation of w_j is written as

$$w_j(t) = \sum_{j \neq i} G_{ji}(q) w_i(t) + \sum_k R_{jk}(q) r_k(t) + \sum_l H_{jl}(q) e_l(t), \quad (1)$$

where q is the delay operator, i.e. $q^{-1} w_j(t) = w_j(t-1)$, and its matrix form is obtained as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t),$$

where $G(q)$ is hollow, $w(t) = [w_1(t), \cdots, w_L(t)]^\top$, $r(t) = [r_1(t), \cdots, r_K(t)]^\top$, and $e(t) = [e_1(t), \cdots, e_p(t)]^\top$. Furthermore, model (1) satisfies the following assumptions:

Assumption 1

- $G_{ji}(q)$ is a stable and proper rational transfer operator if nonzero, and $R_{ji}(q)$ is a stable and proper rational transfer operator;
- $(I - G(q))^{-1}$ is stable and all principal minors of $\lim_{z\to\infty}(I - G(z))$ are non-zero;
- $H(q)$ is monic, proper and minimum-phase when $p = L$; When $p < L$, i.e., rank-reduced noises, $H(q)$ is structured as $H(q) = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$, with H_a square, proper, monic, stable and minimum phase, see (Weerts et al., 2018a) for more details;
- $e(t)$ is white noise vector with covariance matrix $\Lambda > 0$;
- $w(t)$ and $r(t)$ can be measured.

We further define $X(q) \triangleq [R(q) \quad H(q)]$ and call both excitation and noise signals *external signals*. We use *modules* to refer to the transfer functions in $G(q)$.

It can found that a network model is completely specified by $(G(q), R(q), H(q), \Lambda)$. Following (Weerts et al., 2018a,c), a set of such models can be obtained by a rational parametrization of every entry in network matrices $(G(q), R(q), H(q))$ and a parameterization of Λ:

Definition 1. A parameterized network model set for a network of L internal variables, K excitation signals and p noise process is defined as

$$\mathcal{M} = \{(G(q,\theta), R(q,\theta), H(q,\theta)), \Lambda(\theta) | \theta \in \Theta \subseteq \mathbb{R}^n\},$$

with all its elements satisfying Assumption 1. ■

There can also be prior knowledge or users' choices such that certain entries in the network matrices of $\mathcal{M}$ are fixed to zero. Then these entries in the matrices of $\mathcal{M}$ should be zero functions. This structural information of $\mathcal{M}$ can also be encoded by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{W} \cup \mathcal{X}$ is a set of vertices representing the set of all internal signals $\mathcal{W}$ and the set of all external signals $\mathcal{X}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes a set of directed edges. A directed edge exits if and only if the corresponding entry in the matrices of $\mathcal{M}$ is not fixed to zero. For example, $G_{ji}(q,\theta)$ is not a zero function if and only if the directed edge from w_i to w_j exists in $\mathcal{G}$. Then this model set with its structural information is specified by the set $\mathcal{M}$ with its graph $\mathcal{G}$. In this work, we typically assume that the above structural information of a model set is available.

As mentiond in (Weerts et al., 2018a,c), there can also be information that certain entries of the network matrices can be non-parameterized and known, while the other non-zero entries are unknown and parameterized.

For notation, we sometimes use G to denote $G(q,\theta)$, similarly for other functions of θ and q. We denote $T \triangleq (I - G)^{-1} X$, $T_{wr} \triangleq (I-G)^{-1} R$ and $T_{we} \triangleq (I-G)^{-1} H$, while $\Phi_{\bar{v}}(w)$ is the power spectrum of $T_{we}(q)e(t)$.

2.2 Generic identifiability of a single module

Generic identifiability is a property of a network model set. By combining the identifiability concept in (Weerts et al., 2018a) and the concept of genericity in (Bazanella et al., 2017), generic identifiability of module G_{ji} can be defined as follows.

Definition 2. Module G_{ji} in $\mathcal{M}$ is generically identifiable if for almost all $\theta_1 \in \Theta$, it holds that

$$\begin{cases} T_{wr}(q,\theta_1) = T_{wr}(q,\theta_2) \\ \Phi_{\bar{v}}(\omega,\theta_1) = \Phi_{\bar{v}}(\omega,\theta_2) \end{cases} \implies G_{ji}(q,\theta_1) = G_{ji}(q,\theta_2),$$

for all $\theta_2 \in \Theta$. G_{ji} is globally identifiable if the above implication holds for all $\theta_1 \in \Theta$. ■

In the above definition, the notion of almost all means all elements in a set except the elements in a subset of Lebesgue measure zero.

Since T_{wr} and $\Phi_{\bar{v}}$ can typically be identified based on the measured $w(t)$ and $r(t)$, identifiability refers to the situation that a unique module can be found based on T_{wr} and $\Phi_{\bar{v}}$. Following directly from Proposition 1 and Definition 5 of (Weerts et al., 2018a), Definition 2 can be simplified under certain conditions.

Lemma 1. Let $\mathcal{M}$ satisfy one of the following conditions:

- all modules in $G(q,\theta)$ are strictly proper, or
- there is no algebraic loop [1] in $G(q,\theta)$ and $H^\infty(\theta)\Lambda(\theta)H^\infty(\theta)^T$ is diagonal for all θ, with $H^\infty(\theta) := \lim_{z\to\infty} H(z,\theta)$;

then module G_{ji} in $\mathcal{M}$ is generically identifiable if for almost all $\theta_1 \in \Theta$, it holds that

$$T(q,\theta_1) = T(q,\theta_2) \implies G_{ji}(q,\theta_1) = G_{ji}(q,\theta_2), \quad (2)$$

for all $\theta_2 \in \Theta$. G_{ji} is globally identifiable if the above implication holds for all $\theta_1 \in \Theta$. ■

[1] There exists an algebraic loop around node w_{n_1} if there exists a sequence of integers $n_1, \cdots n_k$ such that $G^\infty_{n_1 n_2} G^\infty_{n_2 n_3} \cdots G^\infty_{n_k n_1} \neq 0$, with $G^\infty_{n_1 n_2} := \lim_{z\to\infty} G_{n_1 n_2}(z)$.

In this work, we consider $\mathcal{M}$ that satisfies both the assumptions in Definition 1 and the conditions in Lemma 1. In addition, the model set should also satisfies (Weerts et al., 2018a,c):

- Every parameterized entry in (G, R, H) covers all possible proper transfer functions [2];
- All parameterized transfer functions are parameterized independently.

In this setting, generic identifiability of a single module is related to the maximum number of vertex disjoint paths as defined in Section 2.3 (Weerts et al., 2018c). Before introducing the results in the above paper, we first introduce the important signals used in the results:

- $\mathcal{W}_j$ denotes the set of all in-coming internal signals of $w_j(t)$ through unknown modules;
- $\mathcal{X}_j$ denotes all external signals that do not have unknown directed edge to w_j.

Thus, $\mathcal{X}_j$ also contains all external signals that do not have directed edges to w_j as the corresponding transfer functions are zero. Then the graphical conditions in (Weerts et al., 2018c) are collected here [3]:

Theorem 2. Given a network model set $\mathcal{M}$ and its graph $\mathcal{G}$, the following conditions are equivalent:

(1) module G_{ji} of $\mathcal{M}$ is generically identifiable;
(2) $b_{\mathcal{X}_j \to \mathcal{W}_j} = b_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}} + 1$;
(3) there exists a set $\mathcal{P}$ containing the maximum number of vertex disjoint paths from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ and also a directed path from $\mathcal{X}_j$ to w_i, such that $\mathcal{P}$ and the directed path are vertex disjoint. ■

Condition (2) does not appear explicitly in (Weerts et al., 2018c); however, condition (3) can be phrased equivalently as condition (2). Related graphic results can be found in (Hendrickx et al., 2019) for a different setting where all internal signals are excited and a subset of them is measured. For that same situation, a graphic simplification procedure is developed in (van Waarde et al., 2018) for testing global identifiability of a full network.

Theorem 2 is demonstrated in the following example.

Example 1. Given the network model set in Fig. 1(a), where the target module is G_{41} and all indicated transfer functions are unknown, we have $\mathcal{W}_4 = \{w_1, w_2, w_3\}$ and $\mathcal{X}_4 = \{r_1\}$ such that $b_{\mathcal{X}_4 \to \mathcal{W}_4} = 1$ consisting of one path $r_1 \to w_1$. However, since $b_{\mathcal{X}_4 \to \mathcal{W}_4 \setminus \{w_1\}}$ also consists of one path $r_1 \to w_2$ and thus $b_{\mathcal{X}_4 \to \mathcal{W}_4 \setminus \{w_1\}} = b_{\mathcal{X}_4 \to \mathcal{W}_4} = 1$, G_{41} is not generically identifiable based on Theorem 2. △

The above result allows to analyse generic identifiability for a given situation. However it does not address the synthesis problem: Given a network model set, how to allocate external signals such that module G_{ji} becomes generically identifiable? In this work, we assume that the user can allocate excitation signals. In the next sections,

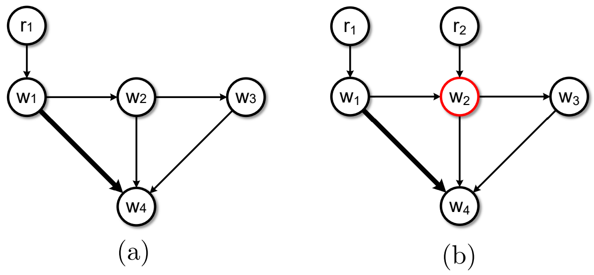

Fig. 1. Generic identifiability of G_{41} is considered (thick line). G_{41} is not generically identifiable in (a) but becomes generically identifiable in (b) if an extra signal u_2 is allocated at w_2.

new analytic results are derived, which lead to several synthesis approaches.

2.3 Notations and concepts

Given $\bar{\mathcal{W}} \subseteq \mathcal{W}$ and $\bar{\mathcal{X}} \subseteq \mathcal{X}$, $T_{\bar{\mathcal{X}}\bar{\mathcal{X}}}$ denotes a submatrix of T with the rows and the columns corresponding to $\bar{\mathcal{W}}$ and $\bar{\mathcal{X}}$ respectively. Throughout the paper, w_i is used to denote both a signal and a vertex in $\mathcal{G}$. For any directed edge $(w_i, w_j) \in \mathcal{E}$, w_i is called an in-neighbor of w_j, and w_j is called an out-neighbor of w_i. For any vertex in $\mathcal{V}$, e.g. w_i, $\mathcal{N}^+_{w_i}$ and $\mathcal{N}^-_{w_i}$ denote the set of all out-neighbors and the set of all in-neighbors of w_i, respectively. A (directed) *walk* in $\mathcal{G}$ from one vertex to another, e.g. from w_i to w_j, is a sequence of vertices and out-going edges starting from w_i to w_j, while a (directed) *path* is a walk without repeating any vertex. The *length* of a directed path is the number of edges in the path, and a single vertex is regarded as a directed path to itself with length zero. In addition, we use $w_i \to w_j$ to denote a directed path from w_i to w_j, and $\mathcal{V}_1 \to \mathcal{V}_2$ denotes paths from some vertices in $\mathcal{V}_1$ to some vertices in $\mathcal{V}_2$. We refer to *internal vertices* as the vertices in a path excluding the starting and the ending vertices, and note that a directed path with length zero or length one does not have any internal vertex.

Two directed paths are *internally vertex disjoint* if they do not share any internal vertex, while they are called *vertex disjoint* if they do not share any vertex, including the starting and ending vertices. If two paths share any common vertex, we say that they *intersect*. Given two subsets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, $b_{\mathcal{V}_1 \to \mathcal{V}_2}$ denotes the maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$.

3. DISCONNECTING SETS FOR GENERIC IDENTIFIABILITY

In this section, we derive a new analytic result based on the concept of disconnecting sets, which shows to be more suitable for synthesis. A vertex set $\mathcal{D}$ is a disconnecting set from a vertex set $\mathcal{V}_1$ to a set $\mathcal{V}_2$ if there is no directed path from $\mathcal{V}_1$ to $\mathcal{V}_2$ when the vertices in $\mathcal{D}$ are removed (Schrijver, 2003). Note that a disconnecting set from $\mathcal{V}_1$ to $\mathcal{V}_2$ may also include vertices in $\mathcal{V}_1 \cup \mathcal{V}_2$. We call a disconnecting set with the minimum cardinality a *minimum disconnecting set*. The duality between vertex disjoint paths and disconnecting sets is explained in Menger's theorem, which is also explored in (Hendrickx et al., 2019).

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[2] within the constraints of Assumption 1 and Lemma 1

[3] The following generic identifiability result holds under the assumption that all nonzero entries in G, R, H are fully parametrized, and that there are no fixed, non-parametrized modules unequal to zero. However the result also holds true when the non-parametrized modules are chosen "generically", i.e. when they do not introduce any particular dependencies in the network model (Cheng et al., 2019b).

Theorem 3. (Menger's theorem (Schrijver, 2003)). Let $\mathcal{V}_1$, $\mathcal{V}_2$ be two subsets of the vertices in a directed graph. The maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ equals the cardinality of a minimum disconnecting set from $\mathcal{V}_1$ to $\mathcal{V}_2$. ■

We illustrate the relevance of the concept of disconnecting sets to our synthesis problem in the following example.

Example 2. Given the network model in Fig. 1(a), $\{w_2\}$ is a disconnecting set from $\{w_1\}$ to the other in-neighbors of w_4, i.e., $\{w_2, w_3\}$. Now, we allocate an extra excitation signal r_2 at w_2, as shown in Fig. 1(b), and we find that $b_{\mathcal{X}_4 \to \mathcal{W}_4} > b_{\mathcal{X}_4 \to \mathcal{W}_4 \setminus \{w_1\}}$, which implies that G_{41} is generically identifiable according to Theorem 2. △

In Example 2, generic identifiability of G_{ji} is achieved when the vertices in a disconnecting set from $\{w_i\}$ to the other in-neighbors of w_j are excited. In the following, we prove that this result holds for the general case. Before proceeding, a graphical result is provided.

Lemma 4. In a simple directed graph, given a set $\mathcal{P}$ of vertex disjoint paths from vertex set $\mathcal{V}_1$ to a vertex set $\mathcal{V}_2$, there exists a set $\mathcal{P}_{new}$ of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ such that $|\mathcal{P}_{new}| = |\mathcal{P}|$, and paths in $\mathcal{P}_{new}$ are internally vertex disjoint with $\bar{\mathcal{V}}$, where $\bar{\mathcal{V}}$ can be either $\mathcal{V}_1$, $\mathcal{V}_2$ or $\mathcal{V}_1 \cup \mathcal{V}_2$.

The above graphical result will later be used to derive the synthesis approach in Algorithm 1. Moreover, based on Lemma 4, a new analytic result for generic identifiability of a single module can be derived.

Theorem 5. Given atork model set $\mathcal{M}$ and igraph$\mathcal{G}$, module G_{ji} is generically identifiable in $\mathcal{M}$ if and only if there exists a set of external signals $\bar{\mathcal{X}}_j \subseteq \mathcal{X}_j$ and a disconnecting set $\mathcal{D}$ from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ such that

$$b_{\bar{\mathcal{X}}_j \to \mathcal{D} \cup \{w_i\}} = |\mathcal{D}| + 1. \tag{3}$$

In the above result, a disconnecting set $\mathcal{D}$ from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ always exists, and one simple example is $\mathcal{W}_j \setminus \{w_i\}$. However, generic identifiability of G_{ji} requires the existence of a special disconnecting set which satisfies equation (3). In addition, note that $\mathcal{D}$ and $\mathcal{W}_j \setminus \{w_i\}$ are not necessarily disjoint, while $\{w_i\}$ and $\mathcal{D}$ have to be *disjoint* to satisfy equation (3).

Theorem 5 can be visualized in Fig. 2, where the paths from w_i and $\bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ intersect with $\mathcal{D}$. Due to (3), we can find a set of vertex disjoint paths consisting of a path $\bar{\mathcal{X}}_j \to w_i$ and multiple paths $\bar{\mathcal{X}}_j \to \mathcal{D}$. As shown in Fig. 2, we can further concatenate the paths $\bar{\mathcal{X}}_j \to \mathcal{D}$ and the paths $\mathcal{D} \to \mathcal{W}_j \setminus \{w_i\}$, which leads to a set of paths $\mathcal{P}$ from $\bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, that are vertex disjoint with the path $\bar{\mathcal{X}}_j \to w_i$. Then based on condition (3) of Theorem 2, G_{ji} is generically identifiable.

Theorem 5 is also closely related to the existing identification method. The considered signals in the disconnecting set can be shown to be the selected signals for the identification of G_{ji} using the direct method (Dankers et al., 2015). There, the internal signals selected in the predictor are the ones that intersect with all loops around the output w_j and all parallel paths from w_i to w_j, where the parallel paths mean all the paths except the edge from w_i to w_j.

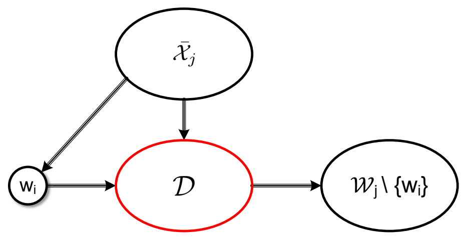

Fig. 2. A graphical visualization of Theorem 5.

Proposition 6. Given $\mathcal{G}$ of a network model set, consider the module G_{ji} and a set of internal signals $\mathcal{D}$ with $w_i \notin \mathcal{D}$. Then $\mathcal{D}$ disconnects from w_i to $\mathcal{W}_j \setminus \{w_i\}$ if and only if $\mathcal{D}$ contains an internal vertex of every parallel path from w_i to w_j and a vertex of every loop around w_j.

The above result shows that the identifiability result in this work and the identification method in (Dankers et al., 2015) coincide from the aspect of signal selection.

Remark 1. A suggestion that results from Theorem 5 is that G_{ji} could be generically identifiable from measured internal signals $\mathcal{D} \cup \{w_i, w_j\}$ only. This is indeed true, as it can be shown that with $\mathcal{U} \triangleq \{w_i\} \cup \mathcal{D}$, and in the case that $\bar{\mathcal{X}}_j$ contains only excitation signals, a consistent estimate of G_{ji} can be obtained using the estimated submatrices $\hat{T}_{j\mathcal{U}}$ and $\hat{T}_{\mathcal{U}\bar{\mathcal{X}}_j}$ of T. This is a direct generalization of the indirect method presented in (Gevers et al., 2018) and extended in (Bazanella et al., 2019), where typically all the in-neighbors of w_j are used instead of $\mathcal{U}$. The formal proof of this result will be presented in future work. △

4. SYNTHESIS APPROACH

Given a network model set with a set of initial external signals $\mathcal{X}_j^0$ that do not have unknown directed edges to w_j, the synthesis problem aims to allocate a minimum number of additional excitation signals $\mathcal{X}_j^a$ such that generic identifiability of a module G_{ji} is guaranteed. For synthesis, it is assumed that if r_k is allocated directly at w_j, its corresponding transfer function R_{jk} is known.

Based on Theorem 5, the main idea of the synthesis approaches is to first compute a disconnecting set $\mathcal{D}$, and then allocate external signals at suitable vertices such that (3) is satisfied.

To reduce the number of allocated excitation signals, a minimum disconnecting set should be used for synthesis; additionally, recall from Theorem 5 that $w_i \notin \mathcal{D}$ is necessary for (3). This implies that a minimum disconnecting set $\mathcal{D}$ subject to $w_i \notin \mathcal{D}$ needs to be found. As standard graphical algorithms for computing minimum disconnecting sets do not take into account any constraint, we redefine the disconnecting set to make standard algorithms applicable.

Lemma 7. Given a network model set $\mathcal{M}$ and its graph $\mathcal{G}$, for any subset $\bar{\mathcal{X}}_j \subseteq \mathcal{X}_j$, a minimum disconnecting set from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ is a minimum disconnecting set $\mathcal{D}$ from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ subject to $w_i \notin \mathcal{D}$. ■

Based on the above approach, a minimum disconnecting set $\mathcal{D}$ from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ can be computed for the synthesis problem, which is an unconstrained problem

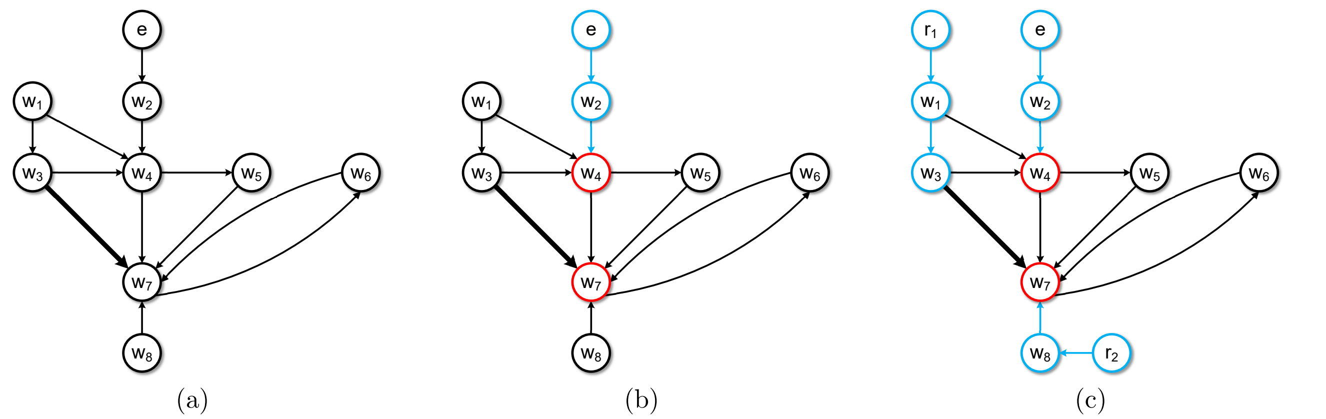

Fig. 3. An example of allocating signals for generic identifiability of G_{73} (thick line) using Algorithm 1. Starting from the network model in (a), a disconnecting set (red vertices) is computed in (b). Since there already exists an external signal e, we only need to add r_1 and r_2 as in (c), which achieves generic identifiability of G_{73}.

and thus can be solved by standard graphic algorithms, e.g. the Ford-Fulkerson algorithm (Schrijver, 2003). Then based on the above lemma and Theorem 5, the following synthesis approach can be derived.

Lemma 8. In a network model set $\mathcal{M}$ with its graph $\mathcal{G}$, given any minimum disconnecting set $\mathcal{D}$ from $\mathcal{N}_{w_i}^+$ to $\mathcal{W}_j \setminus \{w_i\}$, assigning distinct excitation signals to every vertex in $\mathcal{D} \cup \{w_i\}$ leads to generic identifiability of G_{ji} in $\mathcal{M}$. ■

The above result leads to a synthesis approach with simple implementation. However, the method is strictly local in the sense that all excitation signals are directly allocated at w_i and $\mathcal{D}$. In addition, the method does not take into account the initially present external signals $\mathcal{X}_j^0$; however, these signals can also be useful to satisfy equation (3) and thus reduce the number of required allocated signals. Therefore, we present a more comprehensive synthesis approach in Algorithm 1, which makes use of $\mathcal{X}_j^0$ and explores the freedom of allocating additional excitation signals in the graph. Note that $\mathcal{X}_j^0$ can contain both excitation signals and noise signals.

Given a network model set with a target module G_{ji} and a set of pre-existing signals $\mathcal{X}_j^0$, Algorithm 1 first computes a minimum disconnecting set $\mathcal{D}$ from $\{w_i\} \cup \mathcal{X}_j^0$ to $\mathcal{W}_j \setminus \{w_i\}$, and then removes the elements in $\mathcal{D} \cup \{w_i\}$ that are already excited by the existing external signals through vertex disjoint paths. Then it allocates additional signals to excite the remaining vertices in $\mathcal{D} \cup \{w_i\}$ through vertex disjoint paths. The validity of Algorithm 1 is shown as follows.

Theorem 9. In the returned $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$ from Algorithm 1, the target module G_{ji} is generically identifiable. ■

We demonstrate Algorithm 1 by the following example.

Example 3. In the network model in Fig. 3(a), the problem is to allocate excitation signals such that G_{73} becomes generically identifiable. $\mathcal{X}_7^0 = \{e\}$ is the only external signal that is initially present. Firstly, a disconnecting set from $\mathcal{X}_7^0 \cup \{w_3\} = \{w_3, e\}$ to the other in-neighbors of w_7, i.e. $\mathcal{W}_7 \setminus \{w_3\} = \{w_4, w_5, w_6, w_8\}$, is constructed as $\mathcal{D} = \{w_4, w_7\}$, indicated in Fig. 3(b). Based on Theorem 5, generic identifiability of G_{73} requires three vertex disjoint paths from external signals to $\mathcal{D} \cup \{w_3\} = \{w_3, w_4, w_7\}$,

Algorithm 1 Signal allocation for a single module

INPUT: A network model set $\mathcal{M}$ with graph $\mathcal{G}$, a target module G_{ji}, and a set of initial external signals $\mathcal{X}_j^0$ for G_{ji};
OUTPUT: A new model set $\mathcal{M}_{out}$ with its graph $\mathcal{G}_{out}$

1: Compute a minimum disconnecting set $\mathcal{D}$ from $\mathcal{N}_{w_i}^+ \cup \mathcal{X}_j^0$ to $\mathcal{W}_j \setminus \{w_i\}$;
2: Based on Lemma 4, compute a set $\mathcal{P}$ containing the maximum number of vertex disjoint paths from $\mathcal{X}_j^0$ to $\mathcal{D} \cup \{w_i\}$ while the paths are internally vertex disjoint with $\mathcal{D} \cup \{w_i\}$;
3: Let $\bar{\mathcal{D}} \subseteq \mathcal{D} \cup \{w_i\}$ denote all the ending vertices of the paths in $\mathcal{P}$;
4: **if** $|\mathcal{P}| < |\mathcal{D}| + 1$ **then**
5: Find the set $\bar{\mathcal{W}} \subseteq \mathcal{W}$ such that $\mathcal{D}$ is a disconnecting set from $\bar{\mathcal{W}}$ to $\mathcal{W}_j \setminus \{w_i\}$;
6: Build a subgraph $\bar{\mathcal{G}} \subseteq \mathcal{G}$ by removing all vertices in $\mathcal{P}$ and their corresponding edges;
7: Find the set $\mathcal{W}_{exp} \subseteq \bar{\mathcal{W}}$ such that in $\bar{\mathcal{G}}$, there are $|\mathcal{D}| + 1 - |\mathcal{P}|$ vertex disjoint paths from $\mathcal{W}_{exp}$ to $(\mathcal{D} \cup \{w_i\}) \setminus \bar{\mathcal{D}}$;
8: In $\mathcal{G}$, assign distinct excitation signals to every vertex in $\mathcal{W}_{exp}$, which leads to a new model set $\mathcal{M}_{out}$ with a new graph $\mathcal{G}_{out}$;
9: Return $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$;
10: **else**
11: $\mathcal{M}_{out} \leftarrow \mathcal{M}$ and $\mathcal{G}_{out} \leftarrow \mathcal{G}$;
12: Return $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$.
13: **end if**

while $\mathcal{D}$ remains a disconnecting set from the external signals to $\mathcal{W}_7 \setminus \{w_3\}$. Following step 2 in Algorithm 1, we find a path $e \to w_4$ from $\mathcal{X}_7^0$ to $\mathcal{D} \cup \{w_3\}$ (colored blue in Fig. 3(b)). Thus we only need to allocate extra excitation signals to create two vertex disjoint paths to $\{w_3, w_7\}$, which should be vertex disjoint with $e \to w_4$. As in step 5, the potential locations to allocate excitation signals is $\bar{\mathcal{W}} = \{w_1, w_2, w_3, w_4, w_7, w_8\}$, which satisfies that $\mathcal{D}$ remains a disconnecting set from $\bar{\mathcal{W}}$ to $\mathcal{W}_7 \setminus \{w_3\}$. After removing $e \to w_4$, we choose $\mathcal{W}_{exp} = \{w_1, w_8\} \subseteq \bar{\mathcal{W}}$ to be excited. Now there are two vertex disjoint paths from $\mathcal{W}_{exp}$ to $\{w_3, w_7\}$, and the paths are also vertex disjoint with $e \to w_4$, as indicated by the blue paths in Fig. 3(c). Then G_{73} is generically identifiable. △

Even if Algorithm 1 requires fewer signals than the approach in Lemma 8, there is no guarantee for the minimal number of additional signals when $\bar{\mathcal{X}}_j \neq \emptyset$. However, since the number of allocated signals in Algorithm 1 is $|\mathcal{D}| + 1 - b_{\mathcal{X}_j^0 \to \mathcal{D}\cup\{w_i\}}$, the following bound on the number of additional signals required can be derived.

Corollary 10. Given a network model set $\mathcal{M}$ with its graph $\mathcal{G}$. Consider a set of initial external signals $\mathcal{X}_j^0$ that have no unknown directed edge to w_j. Let $\mathcal{D}$ be a minimum disconnecting set from $\mathcal{N}_{w_i}^+ \cup \mathcal{X}_j^0$ to $\mathcal{W}_j \setminus \{w_i\}$, and c denote the number of excitation signals available for allocation. The number of additional excitation signals c is sufficient to make G_{ji} generically identifiable if

$$c \geqslant |\mathcal{D}| + 1 - b_{\mathcal{X}_j^0 \to \mathcal{D}\cup\{w_i\}}.$$

The presented methods can also handle the presence of known modules, which affects the construction of $\mathcal{W}_j$. The consequence of known modules is shown in this example.

Example 4. Consider a network model set with the same topology as the one in Fig. 1, G_{43} and G_{42} are known in the model set, as shown in Fig. 4. Based on the formulation of $\mathcal{W}_4$ in Theorem 2, w_3 and w_2 do not belong to set $\mathcal{W}_4$ now, as we have $\mathcal{W}_4 = \{w_1\}$ in Fig. 4. Thus, in this example, the minimum disconnecting set is an empty set, and generic identifiability of G_{41} can be achieved by a single excitation on w_1, compared to two excitation signals in Fig. 1(b). ■

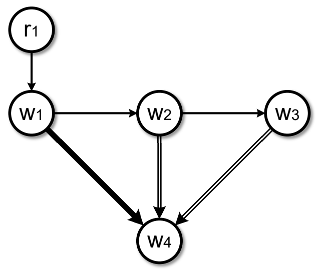

Fig. 4. Generic identifiability of G_{41} with known modules (double-line arrows) can be achieved by only r_1.

5. CONCLUSION

The synthesis problem to achieve generic identifiability of one module in a dynamic network by allocating excitation signals is addressed in this work. A new condition is derived based on the concept of disconnecting set, and then synthesis approaches are developed to allocated excitation signals at the vertices in the disconnecting set. It has been shown that generic identifiability of a single module can be guaranteed by the synthesis approaches.

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Appendix A. PROOF OF LEMMA 4

Proof. We prove the lemma by showing that there always exists a $\mathcal{P}_{new}$ by modifying the paths in $\mathcal{P}$. Let $w_i \to w_j$ be an arbitrary path in $\mathcal{P}$ which contains an internal vertex in $\mathcal{V}$, then we can always replace $w_i \to w_j$ by its subpath which contains a starting vertex in $\mathcal{V}_1$ and an ending vertex in $\mathcal{V}_2$, while the other vertices in the subpath are not in $\mathcal{V}$. This includes the special case that the obtained subpath has no internal vertex. Applying the above modification to all the paths in $\mathcal{P}$ which contain internal vertices in $\mathcal{V}$, we obtain $\mathcal{P}_{new}$, with $|\mathcal{P}_{new}| = |\mathcal{P}|$, in which all the paths are still vertex disjoint.

Appendix B. PROOF OF THEOREM 5

We first introduce a new notation. Given two subsets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ in a graph, $\mathcal{B}_{\mathcal{V}_1 \to \mathcal{V}_2}$ denotes a set containing the maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$.

Proof. To prove the "if" part, we show that if there exists $\bar{\mathcal{X}}_j$ and $\mathcal{D}$ such that equation (3) holds, then G_{ji} is generically identifiable based on condition (3) in Theorem 2, i.e. there exists a directed path from $\mathcal{X}_j$ to w_i and a set of maximum vertex disjoint paths $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$, which is also vertex disjoint with the path from $\mathcal{X}_j$ to w_i.

Equation (3) implies that there is a path $x_i \to w_i$ and a set of vertex disjoint paths from $\bar{\mathcal{X}}_j \setminus \{x_i\}$ to $\mathcal{D}$, denoted by $\mathcal{P}_0$, while $\mathcal{P}_0$ is also vertex disjoint with $x_i \to w_i$. Given the path $x_i \to w_i$, to prove condition (3) in Theorem 2, we only need to show that there exists a set $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$ which is vertex disjoint with $x_i \to w_i$. In the following, we find $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$ based on $\mathcal{P}_0$. We will first introduce a new set of paths $\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ where $\widetilde{\mathcal{X}}_j \triangleq \mathcal{X}_j \setminus \bar{\mathcal{X}}_j$, and then the paths in $\mathcal{P}_0$ and the paths in $\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ will be concatenated such that the target $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$ is obtained. Note that based on Lemma 4, $\mathcal{P}_0$ is chosen such that its paths $\bar{\mathcal{X}}_j \setminus \{x_i\} \to \mathcal{D}$ are internally vertex disjoint with $\mathcal{D}$.

Based on Lemma 4, $\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ is chosen such that all paths in $\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ are internally vertex disjoint with $\mathcal{D} \cup \widetilde{\mathcal{X}}_j$ and $\mathcal{W}_j \setminus \{w_i\}$. Let $\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ be divided into two disjoint subsets, i.e.

$$\mathcal{B}_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}} = \mathcal{P}_{1a} \cup \mathcal{P}_{1b}, \tag{B.1}$$

where

- $\mathcal{P}_{1a}$ contains all paths from $\mathcal{D}$ to $\mathcal{W}_j \setminus \{w_i\}$, and let $\mathcal{W}_{j1}$ be the set of all the ending vertices of the paths;
- $\mathcal{P}_{1b}$ contains all paths from $\widetilde{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and let $\mathcal{W}_{j2}$ be the set of all the ending vertices of the paths in $\mathcal{P}_{1b}$.

Then to construct $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$, a subset $\bar{\mathcal{P}}_0 \subseteq \mathcal{P}_0$ and all paths in $\mathcal{P}_{1a}$ can be concatenated, which leads to a set of walks $\mathcal{P}_2$ from $\bar{\mathcal{X}}_j \setminus \{x_i\}$ via $\mathcal{D}$ to $\mathcal{W}_{j1}$, with $|\bar{\mathcal{P}}_0| = |\mathcal{P}_2| = |\mathcal{P}_{1a}|$. Note that the concatenation is always feasible because based on equation (3), every vertex in $\mathcal{D}$ belongs to a distinct path in $\mathcal{P}_0$. Next, we claim that $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is the desired $\mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$. The claim is true if 1) $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is a set of maximum vertex disjoint paths from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$; 2) $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is vertex disjoint with $x_i \to w_i$. We show the proof for the two statements as follows.

The statement 1) is shown as follows. (i) Firstly, we need to show that the walks in $\mathcal{P}_2$ from concatenation are valid directed paths and thus do not have repeated vertices, i.e. all the paths $\bar{\mathcal{X}}_j \setminus \{x_i\} \to \mathcal{D}$ in $\bar{\mathcal{P}}_0$ and all the paths $\mathcal{D} \to \mathcal{W}_{j1}$ in $\mathcal{P}_{1a}$ are vertex disjoint except at the vertices in $\mathcal{D}$. Assuming that they are not vertex disjoint, a directed path from $\bar{\mathcal{X}}_j \setminus \{x_i\}$ to $\mathcal{W}_{j1}$ will exist without intersecting with $\mathcal{D}$, which contradicts the condition that $\mathcal{D}$ is a disconnecting set and thus proves (i); (ii) Secondly, we show that $\mathcal{P}_2$ and $\mathcal{P}_{1b}$ are vertex disjoint, which suffices to show that $\mathcal{P}_{1a}$ and $\bar{\mathcal{P}}_0$ are vertex disjoint with $\mathcal{P}_{1b}$. It is clear that $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$ are vertex disjoint due to (B.1). Furthermore, $\bar{\mathcal{P}}_0$ and $\mathcal{P}_{1b}$ are also vertex disjoint, because if they are vertex joint, a directed path from $\bar{\mathcal{X}}_j \setminus \{x_i\}$ to $\mathcal{W}_{j2}$ exists without intersecting $\mathcal{D}$, which contradicts the fact that $\mathcal{D}$ is a disconnecting set. (iii) Finally, we prove that $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| = b_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Since $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ contains a set of vertex disjoint paths from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ based on (ii), we have $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| \leqslant b_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Recall that $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| = |\mathcal{P}_{1a} \cup \mathcal{P}_{1b}| = b_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ in equation (B.1), and thus we can prove (iii) by showing the following inequality:

$$b_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}} \geqslant b_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}. \tag{B.2}$$

Let $\mathcal{C}$ be a minimum disconnecting set from $\mathcal{D} \cup \widetilde{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and based on Menger's theorem, it holds that $|\mathcal{C}| = b_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Since all directed paths from $\bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ intersect with $\mathcal{D}$, and $\mathcal{C}$ intersects with all paths from $\mathcal{D}$ to $\mathcal{W}_j \setminus \{w_i\}$, $\mathcal{C}$ is also a disconnecting set from $\bar{\mathcal{X}}_j \cup \widetilde{\mathcal{X}}_j = \mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Then $|\mathcal{C}| \geqslant b_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$ as $\mathcal{C}$ is not necessarily a minimum disconnecting set from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and thus equation (B.2) is also proved due to $|\mathcal{C}| = b_{\mathcal{D} \cup \widetilde{\mathcal{X}}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Consequently, we conclude that $\mathcal{P}_2 \cup \mathcal{P}_{1b} = \mathcal{B}_{\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}}$, i.e. $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is a set of maximum vertex disjoint paths from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$.

We further show the statement 2), which is sufficient to show that $\bar{\mathcal{P}}_0$, $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$ are vertex disjoint with $x_i \to w_i$. Recall that $x_i \to w_i$ is vertex disjoint with $\bar{\mathcal{P}}_0$ based on equation (3). $x_i \to w_i$ is also vertex disjoint with $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$, because if they are joint, there exists a directed path from x_i to $\mathcal{W}_j \setminus \{w_i\}$ or a directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$ without intersecting with $\mathcal{D}$, which contradicts the condition that $\mathcal{D}$ is a disconnecting set. Thus, the $x_i \to w_i$ is vertex disjoint with $\mathcal{P}_2 \cup \mathcal{P}_{1b}$. In conclusion, it has been shown that if equation (3) holds, a set of maximum vertex disjoint paths from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ exists, i.e. set $\mathcal{P}_2 \cup \mathcal{P}_{1b}$, which is also vertex disjoint with a path from $\mathcal{X}_j$ to w_i, i.e. the $x_i \to w_i$ path. Then based on condition (3) of Theorem 2, G_{ji} is generically identifiable. This concludes the proof of the "if" part.

For the "only if" part, given that G_{ji} is generically identifiable, the proof is to find a $\bar{\mathcal{X}}_j$ and $\mathcal{D}$ satisfying equation (3). Based on Theorem 2, if G_{ji} is generically identifiable, there exist a directed path $\mathcal{X}_j \to w_i$ and a set of maximum vertex disjoint paths $\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}$, which is also vertex disjoint with $\mathcal{X}_j \to w_i$. We choose $\bar{\mathcal{X}}_j \triangleq \mathcal{X}_j$ and $\mathcal{D}$ as the minimum disconnecting set from

$\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Then the proof is to show that $\mathcal{D}$ is also a disconnecting set from $\{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$, and the equality in (3) is also satisfied.

To show that $\mathcal{D}$ is also a disconnecting set from w_i to $\mathcal{W}_j \setminus \{w_i\}$, we discuss two cases: (i) If there is no directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$, it is trivial that $\mathcal{D}$ is a disconnecting set from $\mathcal{X}_j \cup \{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$; (ii) If there are paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$, the paths should intersect with $\mathcal{D}$, because if not, a directed path from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ through w_i exists without intersecting $\mathcal{D}$, which contradicts the condition that $\mathcal{D}$ is a minimum disconnecting set from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Based on (i) and (ii), we can conclude that the chosen $\mathcal{D}$ is a disconnecting set from $\mathcal{X}_j \cup \{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$.

Finally, we prove equality (3). Since $\mathcal{D}$ is a minimum disconnecting set from $\mathcal{X}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, we have $b_{\mathcal{X}_j \to \mathcal{D}} = |\mathcal{D}|$, and the vertex disjoint paths counted in $b_{\mathcal{X}_j \to \mathcal{D}}$ can be regarded as subpaths of the paths $\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}$. Since $\mathcal{X}_j \to w_i$ and $\mathcal{X}_j \to \mathcal{W}_j \setminus \{w_i\}$ are vertex disjoint, the paths counted in $b_{\mathcal{X}_j \to \mathcal{D}}$ are also vertex disjoint with $\mathcal{X}_j \to w_i$, which leads to $b_{\mathcal{X}_j \to \mathcal{D} \cup \{w_i\}} = |\mathcal{D}| + 1$. This concludes the "only if" part.

Appendix C. PROOF OF PROPOSITION 6

Proof. We first prove that there exists a directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$ if and only if there exist a parallel path from w_i to w_j or a cycle around the output w_j. Note that due to model (1), $\mathcal{G}$ is a *simple graph*, i.e. there is no self-loop such as (w_i, w_i), and no parallel directed edges from one vertex to another vertex. For "if" part, if there exists a parallel path from w_i to w_j, this parallel path has to intersect with $\mathcal{W}_j \setminus \{w_i\}$. Then we can find a directed path from w_i to one vertex in $\mathcal{W}_j \setminus \{w_i\}$ as a subpath of the parallel path. If a cycle around w_j exists, it will also intersect with $\mathcal{W}_j \setminus \{w_i\}$, and thus the cycle contains a subpath from w_j to one vertex in $\mathcal{W}_j \setminus \{w_i\}$. Linking this subpath and the edge (w_i, w_j) leads to a path from w_i to $\mathcal{W}_j \setminus \{w_i\}$.

For "only if" part, for any directed path from w_i to $w_k \in \mathcal{W}_j \setminus \{w_i\}$, if the path does not contain edge (w_i, w_j), then combining the the path and the edge (w_k, w_j) will create a parallel path. If the path contains (w_i, w_j), then combining the path and the edge (w_k, w_j) while excluding (w_i, w_j) will lead to a cycle around w_j. This concludes the relationship between the parallel paths, the cycles around the output and the paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$.

As seen from the above relationship, the "only if" part is straightforward. For the "if" part, if we collect an internal vertex from each parallel path and a vertex from cycle around the output into $\mathcal{D}$, $\mathcal{D}$ then must disconnect from w_i to $\mathcal{W}_j \setminus \{w_i\}$.

Appendix D. PROOF OF LEMMA 7

We will first show that for any vertex set $\mathcal{C}$ subject to $w_i \notin \mathcal{D}$, $\mathcal{D}$ is a disconnecting set from $\mathcal{N}^+_{w_i} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ if and only if it is also a disconnecting set from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. The "only if" part holds because if $\mathcal{D}$ intersects with all paths from $\mathcal{N}^+_{w_i} \cup \bar{\mathcal{X}}_j$, then it also intersects with the paths from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. For the "if" part, since $w_i \notin \mathcal{D}$ and $\mathcal{D}$ intersects with all the paths from $\{w_i\} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, those paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$ have to intersect with $\mathcal{D}$ at their internal vertices or the ending vertices. Since the first internal vertices of the paths belong to set $\mathcal{N}^+_{w_i}$, then $\mathcal{D}$ is a also disconnecting set from $\mathcal{N}^+_{w_i} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$.

Having the above result, the proposition is proved by showing that a minimum disconnecting set $\mathcal{D}$ from $\mathcal{N}^+_{w_i} \cup \bar{\mathcal{X}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ does not contain w_i, because if it does, it remains a disconnecting set after w_i is excluded, which contradicts the condition that $\mathcal{D}$ is a minimum disconnecting set.

Appendix E. PROOF OF THEOREM 9

Proof. From step 1 to 2 in Algorithm 1, by construction, if $|\mathcal{P}| = |\mathcal{D}| + 1$, equation (3) holds with $\bar{\mathcal{X}}_j = \mathcal{X}^0_j$, and thus the module is generically identifiable in the original model set $\mathcal{M}$.

When $|\mathcal{P}| < |\mathcal{D}| + 1$, based on Theorem 5, we need to allocate extra $|\mathcal{D}| + 1 - |\mathcal{P}|$ signals, such that: (i) there are $|\mathcal{D}| + 1 - |\mathcal{P}|$ vertex disjoint paths from these signals to $(\mathcal{D} \cup \{w_i\}) \setminus \bar{\mathcal{D}}$, and the paths are also vertex disjoint with $\mathcal{P}$; (ii) $\mathcal{D}$ remains a disconnecting set from the added signals to $\mathcal{W}_j \setminus \{w_i\}$.

Then we can find in the algorithm, steps 5 guarantees (ii), and steps 3, 6, 7 and 8 guarantee (i). This concludes that we obtain a new model set with generically identifiable G_{ji} after applying the algorithm.